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CRITERION FOR THE FUNDAMENTALLY MATRIX OF THE SPECIAL DIFFERENTIAL SYSTEMS

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1. Notations and Preliminaries

We consider the differential system

$$x^{'}(t) = (a(t)A_n + b(t)E_n)\, x(t), \tag{1}$$

where $a(t), b(t) \in C_0(J =< t_0, \infty), R)$, $a(t) \neq 0$ for all $t \in J$, E_n is the identity matrix and for the matrix A_n are $a_{i1} = -1$, $a_{1i} = a_{ni} = a_{in} = 1$, if $i = 1, 2, \ldots, n-1$ and the others $a_{ij} = 0$,$j = 1, 2, \ldots, n$,$n \in N, n \geq 3$. We denote

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$$g_1(t,x(t)) = b(t)x_1 + a(t)\sum_{j=2}^{n-1} x_j,$$

$$g_i(t,x(t)) = b(t)x_i + a(t)\left(-x_1 + x_n\right), \quad \text{if} \quad i = 2,3,\ldots,n-1,$$

$$g_n(t,x(t)) = b(t)x_n + a(t)\sum_{j=2}^{n-1} x_j.$$

Here one and only one integral curve of the system (1) passes through every point $(t,x(t)) \in J \times R^n$, because the partial derivatives $\partial g_i/\partial x_j,\ i,j \in \{1,2,\ldots,n\}$ are the continuous functions [6].

We apply to the system (1) the new method for determined the fundamental matrix.

We denote the matrix

$$D_n(t) = a(t)A_n + (b(t) - \lambda(t))E_n. \tag{2}$$

Definition 1. The function $\lambda(t)$ which is the solution of the auxiliary equation $|D_n(t)| = 0$, is called the eigenfunction of the system (1), see [2].

Theorem 2. *Let the function $\lambda(t)$ is the eigenfunction of the matrix system (1), then for every $n \in N, n \geq 3$; $|D_n(t)| = (b(t) - \lambda(t))^n$.*

Proof. According to (2), we can write

$$|D_n(t)| = \begin{vmatrix} z(t) & b_{n-2}(t) & 0 \\ -b_{n-2}^T(t) & z(t)E_{n-2} & b_{n-2}^T(t) \\ 0 & b_{n-2}(t) & z(t) \end{vmatrix},$$

where $z(t) = b(t) - \lambda(t)$ and $b_{n-2}(t) = a(t)(1,1,\ldots,1) \in R^{n-2}$. Moreover

$$|D_n(t)| = \begin{vmatrix} z(t) & b_{n-2}(t) & 0 \\ -b_{n-2}^T(t) & z(t)E_{n-2} & b_{n-2}^T(t) \\ 0 & b_{n-2}(t) & z(t) \end{vmatrix} = \begin{vmatrix} z(t) & b_{n-2}(t) & 0 \\ 0 & z(t)E_{n-2} & b_{n-2}^T(t) \\ z(t) & b_{n-2}(t) & z(t) \end{vmatrix}$$

$$= \begin{vmatrix} z(t) & b_{n-2}(t) & 0 \\ 0 & z(t)E_{n-2} & b_{n-2}^T(t) \\ 0 & 0 & z(t) \end{vmatrix} = z^2(t)\left|z(t)E_{n-2}\right| = z^n(t) = (b(t) - \lambda(t))^n. \quad \square$$

Theorem 3. *For every $n \in N$, $n \geq 3$; $A_n^3 = O_n$.*

Proof. For the matrix A_n it follows

$$A_n = \begin{pmatrix} 0 & I_{n-2} & 0 \\ -I_{n-2}^T & O_{n-2} & I_{n-2}^T \\ 0 & I_{n-2} & 0 \end{pmatrix},$$

where $I_{n-2} = (1, 1, \dots, 1) \in R^{n-2}$, O_{n-2} is the square zero matrix. Denote $o_{n-2} = (0, 0, \dots, 0) \in R^{n-2}$. Then

$$A_n^2 = \begin{pmatrix} 0 & I_{n-2} & 0 \\ -I_{n-2}^T & O_{n-2} & I_{n-2}^T \\ 0 & I_{n-2} & 0 \end{pmatrix} \begin{pmatrix} 0 & I_{n-2} & 0 \\ -I_{n-2}^T & O_{n-2} & I_{n-2}^T \\ 0 & I_{n-2} & 0 \end{pmatrix} = \begin{pmatrix} 2-n & o_{n-2} & n-2 \\ o_{n-2}^T & O_{n-2} & o_{n-2}^T \\ 2-n & o_{n-2} & n-2 \end{pmatrix},$$

$$A_n^3 = \begin{pmatrix} 2-n & o_{n-2} & n-2 \\ o_{n-2}^T & O_{n-2} & o_{n-2}^T \\ 2-n & o_{n-2} & n-2 \end{pmatrix} \begin{pmatrix} 0 & I_{n-2} & 0 \\ -I_{n-2}^T & O_{n-2} & I_{n-2}^T \\ 0 & I_{n-2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & o_{n-2} & 0 \\ o_{n-2}^T & O_{n-2} & o_{n-2}^T \\ 0 & o_{n-2} & 0 \end{pmatrix}.$$

The proof of Theorem 3 is complete. □

2. New Results

Theorem 4. *Let is denote the system (1). If we denote* $P_0 = E_n, P_1 = A_n, \dots, P_{m-1} = A_n^{m-1}$, $m = n-1$ *and*

$$\begin{aligned} q_1(t) &= \exp \int_{t_0}^{t} b(s)ds, \\ q_2(t) &= \int_{t_0}^{t} a(s_1)ds_1 \exp \int_{t_0}^{t} b(s)ds, \\ &\vdots \\ q_m(t) &= \int_{t_0}^{t} \int_{t_0}^{s_{m-1}} \dots \int_{t_0}^{s_2} \prod_{i=1}^{m-1} a(s_i)ds_1 ds_2 \dots ds_{m-1} \exp \int_{t_0}^{t} b(s)ds, \end{aligned}$$

then the fundamentally matrix of system (1) has the form

$$U(t) = q_1(t)P_0 + q_2(t)P_1 + \dots + q_m(t)P_{m-1}, \tag{3}$$

and the general solution of the system (1) is

$$x(t) = U(t)C,$$

where $C = (c_1, c_2, \dots, c_n)^T$ *is a constant vector.*

Proof. According to Theorem 2, for eigenfunctions of system (1) it follows

$\lambda_1(t) = \lambda_2(t) = \cdots = \lambda_n(t) = b(t)$. The functions

$$\begin{aligned} q_1(t) &= \exp \int_{t_0}^{t} b(s)ds, \\ q_2(t) &= \int_{t_0}^{t} a(s_1)ds_1 \exp \int_{t_0}^{t} b(s)ds, \\ &\vdots \\ q_m(t) &= \int_{t_0}^{t} \int_{t_0}^{s_{m-1}} \dots \int_{t_0}^{s_2} \prod_{i=1}^{m-1} a(s_i)ds_1 ds_2 \dots ds_{m-1} \exp \int_{t_0}^{t} b(s)ds, \end{aligned}$$

are the solutions of differential equations (see [1])

$$\begin{aligned} q_1^{'}(t) &= b(t)q_1(t), & q_1(t_0) &= 1, \\ q_2^{'}(t) &= b(t)q_2(t) + a(t)q_1(t), & q_2(t_0) &= 0, \\ &\vdots \\ q_m^{'}(t) &= b(t)q_m(t) + a(t)q_{m-1}(t), & q_m(t_0) &= 0. \end{aligned}$$

We shall prove, that the matrix $U(t) = \sum_{i=1}^{m} q_i(t)P_{i-1}$ is the fundamentally matrix of system (1). Differentiating the equation (3) we get

$$\begin{aligned} U^{'}(t) = \sum_{i=1}^{m} q_i^{'}(t)P_{i-1} &= b(t)q_1(t)P_0 + \sum_{i=2}^{m}(b(t)q_i(t) + a(t)q_{i-1}(t))P_{i-1} \\ &= b(t)\sum_{i=1}^{m} q_i(t)P_{i-1} + a(t)\sum_{i=1}^{m-1} q_i(t)P_i = b(t)U(t) + a(t)\sum_{i=1}^{m} q_i(t)P_i. \end{aligned}$$

According to Theorem 3 it follows $P_m = P_{m-1}P_1 = A_n^m = O_n$. Then $U^{'}(t) = b(t)U(t) + a(t)P_1 \sum_{i=1}^{m} q_i(t)P_{i-1} = (b(t)P_0 + a(t)P_1)U(t)$, hence U is the fundamental matrix of system (1), i.e. the columns of matrix U are the linear independent solutions of differential system (1). The general solution of system (1) has the form [3]

$$x(t) = U(t)C\,,$$

where $C = (c_1, c_2, \dots, c_n)^T$ is a constant vector.

The proof of Theorem 4 is complete. □

3. Example

We consider the differential system

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}' = \begin{pmatrix} b(t) & a(t) & a(t) & 0 \\ -a(t) & b(t) & 0 & a(t) \\ -a(t) & 0 & b(t) & a(t) \\ 0 & a(t) & a(t) & b(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}, \qquad (4)$$

where $a(t), b(t) \in C_0(J =< t_0, \infty), R)$, $a(t) \neq 0$ for all $t \in J$.

By generalized the method of the eigenvalue and eigenvector for the linear differential system with the constant coefficients the integral equations system is expressed and is equipollent with the system (4).

We denote the matrix of the system (4) as follows

$$B(t) = \begin{pmatrix} b(t) & a(t) & a(t) & 0 \\ -a(t) & b(t) & 0 & a(t) \\ -a(t) & 0 & b(t) & a(t) \\ 0 & a(t) & a(t) & b(t) \end{pmatrix}.$$

The function $\lambda(t)$ which is the solution of the equation $\det(B(t)-\lambda(t)E_4) = 0$, where E_4 is the identity matrix, we shall call eigenfunction of system (4).

The vector function $h(t)$ which is the solution of the equation

$$(B(t) - \lambda(t)E_4)h(t) = 0,$$

is called own vector function related to own function $\lambda(t)$ of system (4).

Theorem 5. *The general solution of system (4) has the form* $x(t) = \Big(\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)\Big)C$, *where* $C = (c_1, c_2, c_3, c_4)^T$ *is a constant vector and the vector functions* $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$, $\xi_4(t)$ *is*

$$\begin{aligned} \xi_1(t) &= \Big((1,0,0,0)^T + (0,-1,-1,0)^T \int_{t_0}^t a(s)ds \\ &+ (-2,0,0,-2)^T \int_{t_0}^t \Big(a(s) \int_{t_0}^s a(u)du\Big)\, ds\Big) \exp \int_{t_0}^t b(s)ds, \\ \xi_2(t) &= \Big((0,1,0,0)^T + (1,0,0,1)^T \int_{t_0}^t a(s)ds\Big) \exp \int_{t_0}^t b(s)ds, \\ \xi_3(t) &= \Big((0,0,1,0)^T + (1,0,0,1)^T \int_{t_0}^t a(s)ds\Big) \exp \int_{t_0}^t b(s)ds, \\ \xi_4(t) &= \Big((0,0,0,1)^T + (0,1,1,0)^T \int_{t_0}^t a(s)ds \end{aligned}$$

$$+ \quad (2,0,0,2)^T \int_{t_0}^{t} \left(a(s) \int_{t_0}^{s} a(u)du \right) ds \Big) \exp \int_{t_0}^{t} b(s)ds.$$

Proof. The eigenfunctions of the system (4) are $\lambda_{1234}(t) = b(t)$. We construct matrix sequence

$$P_0 = E_4\,,$$

$$P_1 = (B(t) - b(t)E_4)/a(t) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},\ P_2 = P_1^2 = \begin{pmatrix} -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix},$$

and consider the differential equations

$$q_1^{'}(t) \;=\; b(t)q_1(t), \qquad q_1(t_0) = 1, \tag{5}$$

$$q_2^{'}(t) \;=\; b(t)q_2(t) + a(t)q_1(t), \qquad q_2(t_0) = 0, \tag{6}$$

$$q_3^{'}(t) \;=\; b(t)q_3(t) + a(t)q_2(t), \qquad q_3(t_0) = 0. \tag{7}$$

The solutions of the differential equations (6), (7) and (8) are functions

$$q_1(t) \;=\; \exp \int_{t_0}^{t} b(s)ds, \tag{8}$$

$$q_2(t) \;=\; \int_{t_0}^{t} a(s)ds \exp \int_{t_0}^{t} b(s)ds, \tag{9}$$

$$q_3(t) \;=\; \int_{t_0}^{t} \left(a(s) \int_{t_0}^{s} a(u)du \right) ds \exp \int_{t_0}^{t} b(s)ds. \tag{10}$$

We shall prove, that the matrix

$$U(t) = q_1(t)P_0 + q_2(t)P_1 + q_3(t)P_2 \tag{11}$$

is the fundamental matrix of the system (4). Differentiating the equation (11), we obtain
$U^{'}(t) = q_1^{'}(t)P_0 + q_2^{'}(t)P_1 + q_3^{'}(t)P_2 = b(t)q_1(t)P_0 + (b(t)q_2(t) + a(t)q_1(t))P_1 + (b(t)q_3(t) + a(t)q_2(t))P_2 = b(t)(q_1(t)P_0 + q_2(t)P_1 + q_3(t)P_2) + a(t)(q_1(t)P_1 + q_2(t)P_2) = b(t)U(t) + a(t)(q_1(t)P_1 + q_2(t)P_2 + q_3(t)P_3)$, where $P_3 = P_2P_1 = 0$.

Since $U^{'}(t) = b(t)U(t) + a(t)P_1(q_1(t)P_0 + q_2(t)P_1 + q_3(t)P_2) = (b(t) + a(t)P_1)U(t) = B(t)U(t)$, the matrix U is the fundamental matrix of system (1), i.e. the columns of matrix U are the linear independent solutions of differential system (4). The general solution of system (4) has the form

$$x(t) = U(t).C\,,$$

where $C = (c_1, c_2, c_3, c_4)^T$ is a constant vector. The proof is complete. □

The authors have investigated system of quasilinear differential equations with matrix $A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & a_{23} & 0 \end{pmatrix}$, where $a_{ij} \neq 0$, $1 \leq i < j \leq 3$ are real numbers (see [4]). In the paper [5] is investigated system of linear differential equations with matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. The asymptotic and oscillatory properties of solutions of the non-linear differential systems are investigated in the paper [10].

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