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GPS-AIDED INERTIAL NAVIGATION ALGORITHMS: NEW APPROACHES

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Abstract: Use of GPS-aided inertial navigation systems is continuously increasing thanks to the remarkable evolution of sensor technology and data processing methods. Several algorithms are developed to realize an INS/GPS (Inertial Navigation System/Global Positioning System) software to post-process navigation data. Navigation equations are analytically solved. For data noise reduction a wavelet denoising approach, with multilevel decomposition and successive synthesis of terms of higher scale only, is used. Two different statistical data fusion algorithms which integrate INS data with GPS measures are applied: Kalman filtering plus smoothing and a batch or geodetic solution. Both procedures allow to integrate INS raw data with estimated positions of a tern of GPS antennas mounted on a vehicle. The geodetic solution is applied for the first time to inertial navigation real data. Some tests are performed to compare the two solutions.

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1. Introduction

Inertial navigation stands for the determination of position and attitude of a vehicle with respect to some known reference system (frame), by using measures from a set of inertial sensors, usually three gyroscopes and three accelerometers mounted along the axes of an orthogonal Cartesian frame, which is fixed to the vehicle (body frame or b-frame). Gyros measure rotation rate, while accelerometers measure accelerations, both with respect to an inertial frame; measured data are expressed in b-frame. The set of inertial sensors, is named SDIMU (StrapDown Inertial Measurement Unit). An *inertial navigation system* (INS) consists of a SDIMU plus a processor which estimates position and attitude. Solutions can be both real time or post-processed; this work investigates on post-processed solutions.

2. Navigation Equations

Inertial navigation is essentially based on integration of accelerations measured at different epochs, solving a differential equation system that links measured accelerations with second derivatives of the position, the so-called *navigation equations*. In a SDIMU, measures are expressed in a reference frame, the b-frame, which changes with time. It is preliminarily necessary to estimate the rotation between the b-frame and a known system, to which the accelerations to be integrated must be referred. Its determination implies the solution of a further differential equation system. It has been chosen to solve the equations in the inertial frame (i-frame) and Earth Centered-Earth Fixed (ECEF) system (e-frame), and to transform the results in a local system. In this case in fact some equations decouple. Usually for survey applications the solution is numerically computed, in this work the differential equations have been analytically solved on each sampling interval (as in Cazzaniga [2]).

2.1. Attitude Determination

Given two frames, i stationary and b rotating with respect to i, the following relation holds

$$\dot{R}_b^i = \lim_{\delta t \to 0} \frac{R_b^i\,(t + \delta t) - R_b^i\,(t)}{\delta t} \tag{1}$$

which implies

$$\dot{R}_b^i = R_b^i \Omega_{ib}^b \,, \tag{2}$$

where

$$\Omega_{ib}^b = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{3}$$

is the skew-symmetric matrix of the angular rates of b-frame with respect to i-frame expressed in b-frame:

$$\omega_{ib}^i = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \alpha}{\partial t} & \dfrac{\partial \beta}{\partial t} & \dfrac{\partial \gamma}{\partial t} \end{pmatrix}. \tag{4}$$

Relation (2) can be expressed in terms of quaternion q_b^i also, and in this case the equation becomes linear

$$\dot{q}_b^i = \frac{1}{2} A q_b^i = B q_b^i \,, \tag{5}$$

where $\omega_{ib}^b = \begin{bmatrix} 0 & \omega_{ib}^b \end{bmatrix}^T$ is the quaternion form of the vector of angular rates (4) and

$$A = \begin{bmatrix} 0 & \omega_1 & \omega_2 & \omega_3 \\ -\omega_1 & 0 & \omega_3 & -\omega_2 \\ -\omega_2 & -\omega_3 & 0 & \omega_1 \\ -\omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

We now introduce the hypothesis that the components of ω_{ib}^b are constant on the integration interval. The hypothesis is acceptable if this interval, equal to the sampling interval, is little enough with respect to the dynamic of the system. For terrestrial applications a sampling frequency of the order of a hundred Hertz is considered sufficient. Under this hypothesis, system (5) can be solved like a differential equation system with constant coefficients.

2.2. Determination of Position

Now, we rotate the measured accelerations in the e-frame:

$$a_{oe} = q_i^e q_b^i a_{ob} \overline{q}_b^i \overline{q}_i^e = q_b^e a_{ob} \overline{q}_b^e. \tag{6}$$

Because the Earth has a uniform rotation rate, we can write

$$\ddot{x}_e = a_{oe} - 2\Omega_e \dot{x}_e - \Omega_e^2 x_e - g_e. \tag{7}$$

a_{oe} are specific forces measured by accelerometers expressed in e-frame, $2\Omega_e \dot{x}_e$ is the Coriolis acceleration, $\Omega_e^2 x_e$ is the centrifugal acceleration and g_e is the gravity acceleration. Usually a unique term $\gamma_e(x_e) = \Omega_e^2 x_e + g_e$ is considered (Heiskanen et al [6]), whose potential is divided into two parts $W = U + T$. U is called normal component and T the anomalous one. The approximation of the gravitational field with the normal gravity field, neglecting the gravity anomaly, is usually sufficient for terrestrial applications. Therefore equation (7) becomes

$$a_e = a_{oe} - 2\Omega_e v_e - \gamma_e. \tag{8}$$

We introduce the hypothesis that also these specific forces and γ_e are constant in the integration interval. Again, such simplification is valid only if the sampling interval is sufficiently little with respect to the system dynamics. In this case it is possible to decouple the first two equations from the last one. The solution of the resulting system is trivial.

The solution of equation (8), like the solution of the gyro equations (5), is a linear equation and describes a linear dynamic system.

3. IMU/GPS Coupling

Since an INS integrates differential equations containing real data, systematic errors lead to velocity and position errors increasing with the integration time. For such a reason INS is not suited to autonomously navigate with a good precision for long periods, therefore it is often coupled with different instruments. This configuration is named *aided navigation*. At present GPS is the most used aiding system in outdoor applications, obtaining a coupled system named INS/GPS (or IMU/GPS). Aided inertial navigation systems utilize independent and complementary systems: one instrument (INS) supplies high accuracy estimations for a short-period, while the other instruments is used for long-period stability.

Data fusion from different sensors requires appropriate statistical methods. In an INS/GPS system the Kalman filter is widely used, often followed by a Kalman smoother in post-processed applications. The so called geodetic batch solution (Albertella et al [1]) is also applied in this work.

3.1. Kalman Filter and Smoother

Kalman filter and smoother allow for the estimation of variables from measures of connected quantities. It is based on the hypotheses that the steady state of the system at a single epoch depends only on the state of the system at the previous epoch and that this dependence is linear. Kalman estimators are optimal in the sense of Wiener-Kolmogorov, they have minimum variance and are normally distributed. Optimality however is grant only as long as assumptions of the model are valid. The solution is stable if observable and check conditions are satisfied (Cazzaniga et al [3]). The main advantage of the method is that the dimension of the involved matrices depends only on the number of measures and parameters of a single epoch and, at most, also of the subsequent epoch.

As it is well-known (Strang and Borre [9]), the steady state equation must be written, which expresses the relation between parameters of two successive epochs and represents the mathematical model chosen to describe the physical phenomenon; the measurement equation, is also needed, in order to link the parameters value and the measurable quantities. The steady-state equations are represented for this problem by the solution of (8) and (5) discretized in time.

3.2. Geodetic Solution

Kinematic data processing is usually performed using Kalman filtering not only to have a real time estimate but also because other solutions require an excessive computation load. Recently in Albertella et al [1], an alternative approach, called *geodetic* or *batch solution*, has been proposed. It basically consists in the application of Least Squares (LS) algorithm, implemented in a way that remarkably reduces the numerical effort.

This method does not use a recursive procedure, but it takes into account all observations $Y_{1,\dots T}$ in one step, in order to estimate all state parameters $X_{1,\dots T}$.

The LS solution can be represented as described in Albertella et al [1]:

$$\hat{\omega} = C^{\omega}(D^{+})^{-1}M^{+}(MD^{-1}C^{\omega}(D^{+})^{-1}M^{+} + C^{\varepsilon})^{-1}Y. \qquad (9)$$

The numerical effort in computing (9) is mainly given by the inversion of matrix $(MD^{-1}C^{\omega}(D^{+})^{-1}M^{+} + C^{\varepsilon})$ (or solution of the corresponding linear system), that is a full matrix with dimension equal to the product of the total number of observations by the total number of epochs (equal to the total number of measurements).

However, equation (9) can be rewritten as

$$\hat{\omega} = D(D^{+}W^{\omega}D + M^{+}W^{\varepsilon}M)^{-1}M^{+}W^{\varepsilon}Y. \qquad (10)$$

The normal matrix of (10) has dimension equal to the number of observations by number of epochs, while the normal matrix in (9) has dimension equal to the number of observations by the number of parameters, so that former can be smaller than the latter. On the other hand, the normal matrix in (9) is full, while that in (10) is block-banded with bandwidth equal to twice the dimension of the vector X.

The solution is given by Albertella et al [1]:

$$\hat{X} = (D^{+}W^{\omega}D + M^{+}W^{\varepsilon}M)^{-1}M^{+}W^{\varepsilon}Y \quad (11)$$

and the covariance matrix is

$$C_{ee} = (D^{+}W^{\omega}D + M^{+}W^{\varepsilon}M)^{-1}. \quad (12)$$

The developed GPS-aided solution with Kalman filter is not substantially different from the geodetic one, as this simply needs a "re-organization" of steady-state and measurement equations. To determine the geodetic solution, the same steady-state and measurement equations used for Kalman filter solution are required.

According to this method, the problem becomes a LS estimation, where the normal matrix is a band-blocked one with band amplitude equal to twice the dimension of vector x_i. As it is well known, this problem can be efficiently solved using Cholesky decomposition. Considering the variance-covariance matrix, the inverse of a band matrix is in general a full matrix. The lemma of Erisman-Tinney (Erisman and Tinney [5]) however assures that coefficients inside the band of an inverse matrix can be obtained without knowing coefficients outside the band. Correlation of a parameter set is obtainable and correlation among parameters of one epoch with those of the previous or successive epoch can also be calculated. These correlation terms are not obtainable by Kalman filtering.

The numerical effort is strictly comparable with that of Kalman filtering followed by Kalman smoothing ($2n$ steps, n = band dimension). In this case the memory storage will be equal to Tdb, where b is the band amplitude and d is the the dimension of the vector x_i. Regarding the solution of the normal system, the number of the operations performed by the computer will be of the order of Tdb^2. A computer can allocate memory for a single variable till a maximum limit in the dimension, depending on its characteristics. This value for a last generation elaborator is fairly high to allow for management of few hours sessions. For instance we consider the case of navigation equations that integrate accelerations: there are 9 parameters to be estimated for each observation epoch. In case of a 100 Hz sampling there are 6000 observations each minute, for a total of 54000 parameters each minute. The band amplitude of the normal matrix will be in this case equal to 18. A computer with a 1 Gb

RAM allows to allocate in *Matlab* in a unique matrix the memory necessary for more than four hours of observations.

4. Noise Reduction

Inertial data, like all real measures, have an error component overimposed to true signal, which degrades the estimates. This error is made of a systematic component, which causes solution drift and can be eliminated using models and decoupling systematic effects, and an accidental one, which cannot always be treated as white noise: sometimes it presents one or more dominant frequencies due to non completely random causes (e.g. vehicle vibration).

As well known (Kaiser [8]), continuous wavelet transform of a signal with finite energy $f(t)$ is given by

$$W(s,p) = \int_{-\infty}^{+\infty} f(t) \cdot \bar{\psi}_{s,p}(t)\, dt$$

and allows for signal decomposition (*analysis*). This computation is usually carried out stepwise varying the parameters s and p, obtaining a series of *wavelet coefficients*, which supply the wavelet decomposition of signal.

In case of orthonormal wavelets, the signal can be theoretically reconstructed by summation of base functions weighted with the coefficients of its wavelet transform

$$f(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} W(j,k)\psi_{j,k}(t).$$

Practically, the reconstruction is only possible at a particular level of description.

Wavelets have a band-pass spectrum, so, from Fourier theory, a set of wavelets at different scale factor can be seen as band-pass filter banks of different dimension and position in spectrum. These filters can be constructed in such a way that the spectrum is broken down into frequency ranges. Usually the signal spectrum is divided into two parts (the threshold is chosen by specific algorithms). Iterating this procedure we obtain a multilevel decomposition: at each step only high scale components are decomposed in order to separate high frequency noise and signal due to the motion of the vehicle. Now the synthesis of $f(t)$ using only high scale coefficients obtained from each level implies a noise reduction stronger and stronger increasing the decomposition level.

An important advantage of this procedure is that it is applicable to a non-stationary signal (Cazzaniga [2]), particularly to establish different thresholds during periods at rest, besides it is very effective for bias estimation (De Agostino [4]). Systematic errors in the observations are more evident after applying the wavelet denoising procedure. This simplifies the subsequent step of bias reduction. The choice of the decomposition level has to be carefully done, a too high level can even occur in the range of spectrum where the signal of the motion is present, significantly modifying it. Experimentally, we considered appropriate for our application a four level decomposition.

5. Tests

The algorithms to post-process IMU/GPS data, described in the previous sections, were implemented in *Matlab*. This software works with data of three GPS antennas mounted on the roof of a vehicle, besides GPS-only vehicle positions must be previously estimated by an independent commercial software. To evaluate the performances of the implemented software and of the used sensor, some tests with the mobile mapping vehicle of the University of Parma were carried out; the vehicle was equipped with two photogrammetric cameras, three GPS and a low-cost IMU.

Some tests were organized to compare the performances of the geodetic and Kalman approach. The estimated parameters are, as expected, comparable. Covariance matrices are more realistic in the case of batch estimate, since they take into account the principal temporal correlation among the estimates. The processing time is lower with the geodetic procedure. The Kalman estimate allows (at least in theory) to process a survey unlimited in time, while the geodetic estimate allows to process a survey lasting consecutively more than 4 hours. When the problem requires a real time solution, only the Kalman filter (without smoother) can be used, but in case of a post-processed solution, like for example for land applications of kinematic surveys, the geodetic solution is potentially very useful.

6. Conclusion

The main goal of the work here briefly presented is the development of methods to perform autonomous and GPS-aided navigation. We have defined a navigation procedure to analytically solve the problem, studied a wavelet denoising procedure and applied for the first time the geodetic solution to real data obtaining promising results.

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