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A SPECTRAL METHOD FOR RAREFIED GAS DYNAMICS PROBLEMS IN CYLINDRICAL GEOMETRY

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Abstract: A classical spectral approach based on Legendre polynomials is used to solve an integral equation required to describe rarefied gas flows in a cylindrical tube. The formulation is based on the BGK kinetic model for solving the Poiseuille and thermal creep problems.

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1. Introduction

In the field of the rarefied gas dynamics (RGD) problems, modeled by the linearized Boltzmann equation and by kinetic equations [6], solutions for plane channel problems have been extensively studied [6], [18] over the years. In addition to numerical methods [16], analytical approaches have been proposed and used to solve many problems [8], [7], in an accurate form. For instance, recently, an analytical approach [4] has been used to solve in a an unified manner a wide class of problems in plane geometry, based on several kinetic models [12], [13].

Within the context of deterministic approaches, complex and multidimensional geometries, however, still remain a challenge in our point of view, particularly for analytical tools, due to the complexity of the associated mathematical models in this area. Classical approaches developed for the integro-differential form of the equations do not apply in a straightforward way to those geometries. In this sense, one of the possible ways to be investigated could be formulations in terms of the integral form of the equations [18].

In cylindrical geometry, for a particular class of problems, a well known transformation proposed by Mitsis [11] allows to relate the integral form of the kinetic equation, in cylindrical geometry, to slab problems. Some problems have been successfully solved in this manner [14]. However, its application is restricted to some cases.

Searching for alternative analytical approaches for providing closed form solutions for the integral form of kinetic model equations in RGD cylindrical problems, which might be applied to a more general class of problems, we develop in this work a classical spectral solution based on Legendre polynomials. Spectral methods [15], in particular this technique [3], were already successfully applied to other class of important integral equations in this field, and that was one of the reasons which lead us to choose this approach.

The integral equation we consider here, to describe, according to the BGK model [5], the flow of a rarefied gas in a cylinder of radio R, can be written as [14], [2]

$$Z(r) = \int_0^R tZ(t)\mathcal{K}(t \to r)\mathrm{d}t + S(r), \tag{1}$$

for $r \in [0, R]$. In regard to equation (1),

$$\mathcal{K}(t \to r) = \frac{2}{\pi^{1/2}} \int_0^\infty \mathrm{e}^{-u^2} F_0(t/u, r/u) \frac{\mathrm{d}u}{u^2} \tag{2}$$

with

$$F_0(t/u, r/u) = \begin{cases} I_0(t/u)K_0(r/u), & t \in [0, r]\,, \\ K_0(t/u)I_0(r/u), & t \in [r, R]\,, \end{cases} \tag{3}$$

where we use $I_n(x)$ and $K_n(x)$ to denote the modified Bessel functions of the first and second kind [1], respectively. Still, $S(r)$ is a specified inhomogeneous source term. Particularly, for the two problems we solve in this work, following previous works [14], [10], [17] we write

$$S_P(r) = \frac{1}{2}\pi^{1/2}, \tag{4}$$

for the Poiseuille flow problem and

$$S_T(r) = R\int_0^\infty u\mathrm{e}^{-u^2} K_1(R/u)I_0(r/u)\mathrm{d}u \tag{5}$$

for the thermal-creep problem.

Quantities we are interested on evaluating, for the gas, are written in terms of the unknown function $Z(r)$. For the Poiseuille flow problem, we express [14], [10], [17] the macroscopic velocity profile as

$$q_P(r) = \pi^{-1/2} Z_P(r) - \frac{1}{2} \tag{6}$$

and the flow rate

$$Q_P = \frac{4}{R^3}\int_0^R q_P(r) r\mathrm{d}r. \tag{7}$$

The same quantities are defined, for the thermal-creep problem, as

$$q_T(r) = \pi^{-1/2} Z_T(r) - \frac{1}{4} \tag{8}$$

and

$$Q_T = \frac{4}{R^3}\int_0^R q_T(r) r\mathrm{d}r. \tag{9}$$

2. The Development

First of all, for computational purposes [14], we write

$$\hat{I}_0(x) = I_0(x)\mathrm{e}^{-x}, \tag{10}$$

$$\hat{K}_0(x) = K_0(x)\mathrm{e}^{x}, \tag{11}$$

such that, from now on, in equations (2) and (3), we consider

$$F_0(t/u, r/u) = \begin{cases} \mathrm{e}^{(t-r)/u}\hat{I}_0(t/u)\hat{K}_0(r/u), & t \in [0, r]\,, \\ \mathrm{e}^{(r-t)/u}\hat{K}_0(t/u)\hat{I}_0(r/u), & t \in [r, R]\,. \end{cases} \tag{12}$$

We then start by expressing the solution of equation (1), in terms of Legendre polynomials, as

$$Z(r) = \sum_{\alpha=0}^{L} a_\alpha P_\alpha \left(\frac{2r}{R} - 1 \right) , \tag{13}$$

where the constants $\{a_\alpha\}$ are to be determined. To find the coefficients $\{a_\alpha\}$ required in equation (13), we substitute equation (13) into equation (1), multiply the resulting equation by

$$r P_l \left(\frac{2r}{R} - 1 \right) , \tag{14}$$

for $l = 0, 1, 2, \cdots, L$, and integrate over r to obtain a system of linear algebraic equations

$$\sum_{\alpha=0}^{L} a_\alpha \left[A_{l,\alpha} - \{ B_{l,\alpha} + C_{l,\alpha} \} \right] = D_l \, , \tag{15}$$

where

$$A_{l,\alpha} = \int_0^R r P_l \left(\frac{2r}{R} - 1 \right) P_\alpha \left(\frac{2r}{R} - 1 \right) \mathrm{d}r, \tag{16}$$

$$B_{l,\alpha} = \int_0^R \int_0^r rt P_l \left(\frac{2r}{R} - 1 \right) P_\alpha \left(\frac{2t}{R} - 1 \right) \mathcal{K}(t \to r) \mathrm{d}t \mathrm{d}r, \tag{17}$$

$$C_{l,\alpha} = \int_0^R \int_r^R rt P_l \left(\frac{2r}{R} - 1 \right) P_\alpha \left(\frac{2t}{R} - 1 \right) \mathcal{K}(t \to r) \mathrm{d}t \mathrm{d}r, \tag{18}$$

$$D_l = \int_0^R S(r) r P_l \left(\frac{2r}{R} - 1 \right) \mathrm{d}r. \tag{19}$$

Although the approach lead us to a symmetric system, which reduces the numerical work involved, due to the singular behavior of the kernel, a good and precise evaluation of the above integrals is not an easy task.

3. Computational Aspects and Numerical Results

In this approach, we have some basic approximation parameters, such as $L+1$, the number of basis functions used in equation (13), which is also related to the size of the linear system to be solved. In general, we mapped all the integration intervals to allow us the use of a typical Gauss-quadrature scheme. We used N_r, N_t and N_u number of Gauss points to evaluate equations (16) to (19), to define the entries of the linear system given by equation (15). However, a complete different approach has to be defined in order to deal with the singular behavior of equation (2), when $|r-t|$ becomes smaller (typically less than 0.05). While a simple (multiple) domain division, around the singularity, can be used for dealing with this issue [9], we found that special functions can be added and subtracted (keeping the expression unchanged) to equation (2) to re-define a kernel more amenable to the numerical evaluation [7], and, mainly, to reduce significantly the computational time. One of our choices was, firstly, to subtract from the original kernel, equation (2), the integral

$$S_1(r,t) = \frac{2}{\pi^{1/2}} \int_0^\infty F_0(t/u, r/u) \frac{du}{u^2} \tag{20}$$

which evaluation can be expressed, using *Maple*, in terms of the complete elliptic integrals of the first time, given in [1]. Secondly, after a change of variable, we add

$$S_2 = \frac{2}{\pi^{1/2}} \int_0^1 \mathrm{e}^{-s} \hat{K}_0(s)\, ds, \tag{21}$$

such that the numerical evaluation of equation (2) is done in terms of

$$\mathcal{K}(r \to t) = \frac{2}{\pi^{1/2}} \int_0^1 \Big\{ g(s) F_0\Big(t/u(s), r/u(s)\Big) + \mathrm{e}^{-s} \hat{K}_0(s) \Big\}\, ds + S_1(r,t) - S_2. \tag{22}$$

Here

$$u(s) = \frac{1}{s} - 1 \quad \text{and} \quad g(s) = \frac{\mathrm{e}^{-u(s)^2} - 1}{s^2(1-s)^2}. \tag{23a,b}$$

In this way, we found agreement in 5 to 6 digits with the results obtained via Mitsis transformation [14], using $L = 60$, $N_r = 260$, $N_t = 180$ and $N_u = 80$. After solving equation (15), we susbtitute equation (13) in equations (6) and (8) to find the velocities. Equations (7) and (9) are evaluated by Gauss quadrature.

4. Concluding Remarks

A classical spectral approach was used to develop a closed form solution for an integral equation in the rarefied gas dynamics field. The fact of having successfully used the same approach in earlier works [3] in this field, may lead us to treat new problems, particularly when the Mitsis transformation [11] can not be used. In subsequent papers we also describe the use of another spectral method, based on a collocation scheme, which has been shown to be efficient.

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