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LOWER-SEMICONTINUITY AND OPTIMIZATION OF CONVEX FUNCTIONALS

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Abstract: The result that we treat in this article allows to the utilization of classic tools of convex analysis in the study of optimality conditions in the optimal control convex process for a Volterra-Stietjes linear integral equation in the Banach space $G([a,b],X)$ of the regulated functions in $[a,b]$, that is, the functions $f:[a,b]\to X$ that have only descontinuity of first kind, in Dushnik (or interior) sense, and with an equality linear restriction. In this work we introduce a convex functional $L_{\beta,f}(x)$ of Nemytskii type, and we present conditions for its lower-semicontinuity. As consequence, Weierstrass Theorem garantees (under compacity conditions) the existence of solution to the problem $\min\{L_{\beta,f}(x)\}$.

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1. Introduction

The lower-semicontinuity is a very important notion that plays an important

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role in optimizing convex functionals, that appears in several classic applications. The result that we deal in this article allows to the utilization of classic tools of Convex Analysis in the study of optimality conditions in problems with equality restrictions. We introduce a convex functional of integral Nemytskii type, and we present conditions for its lower-semicontinuity. Given a Banach space X, X^* stands for its dual (the space of all bounded linear functionals on X). We denote by $G([a,b],X)$ the Banach space (with the convergent uniform norm) of the regulated functions in $[a,b]$, that is, the functions $f:[a,b]\to X$ that have only descontinuity of first kind, and by $SV_a([a,b],L(X,Y))$ the space of functions α of bounded semivariation, that is, $SV([a,b])[\alpha]=\sup\{SV(d)[\alpha]\ /\ d\in D\}<\infty$ and $SV(d)[\alpha]=\sup\{\|[\alpha(t_i)-\alpha(t_{i-1})]\ x_i\|\ /\ x_i\in X,\ \|x_i\|\le 1\}$, such that $\alpha(a)=0$ (in particular

$$SV_a([a,b],X^*)=BV_a([a,b],X^*)),$$

see [4]. $SV_a([a,b],L(X))$ becomes a Banach space when endowed with the norm $\|\alpha\|=SV[\alpha]$. The integrals in the Dushnik (or interior) sense that appears are defined, when there exists the limit, by

$$\int_a^b \cdot\, d\,\alpha(t)\cdot f(t)=\lim_{d\in D}\sum_{i=1}^{|d|}[\alpha(t_i)-\alpha(t_{i-1})]\cdot f(\xi_i^{\cdot}) \tag{1}$$

where D is the set of all divisions $d: a=t_0<t_1<...<t_n=b$ of $[a,b]$ ($n=|d|$ is the order of division d) and $\xi_i^{\cdot}\in]t_{i-1},t_i[$. The oscilation of f on $[a,b]$ corresponding to division d will be defined by $\omega_d^{\cdot}(f)=\sup\{\omega_i^{\cdot}(f)\ /\ i=1,2,...,|d|\}$, where $\omega_i^{\cdot}(f)=\sup\{\|f(t)-f(s)\|\ /\ s,t\in]t_{i-1},t_i[\}$. If $\alpha\in SV([a,b],L(X))$ then the integral exists. We consider here the linear evolutive process described by the linear Volterra-Stieltjes integral equation

$$x(t)-x_0+\int_a^t \cdot\, d_s\ K(t,s)\cdot x(s)=u(t),\ t\in[a,b]\,, \tag{K}$$

where the kernel $K\in G_0^{\sigma}\cdot SV^u([a,b]\times[a,b],L(X))$, that is, K is simply regulated as a function of the first variable with $K(t,t)=0$ and of uniformly bounded semivariation as a function of the second variable, and subject to the linear constraint

$$F_\alpha[x]=\int_a^b \cdot\, d\,\alpha(s)\cdot x(s)=0\,. \tag{F_α}$$

Both states x and parameter control u are selected in $G([a,b],X)$. Many examples of systems in mathematical analysis are instances of this system, for example, Stieltjes integral equations, Volterra integral equations, linear delay differential equations, functional equations, impulsive equations. We use the notation $x_u(s)$ to the solution of $(K)+(F_\alpha)$ associate to u in the sense of

Theorem 3.4, [6]. In other words,

$$x_u(t) = u(t) + R(t,a)\cdot[x_0 + u(a)] - \int_a^t \cdot\, d_s\, R(t,s)\cdot u(s)\,, \quad t\in[a,b], \qquad (\rho)$$

where $R \in G_I^\sigma \cdot SV^u$ ($R(t,t) = I_X$) is the resolvent (unique) of K. The properties of the notions given here can be found in [4]. With the support of a representation theorem in $[G^-([a,b],X)]^*$, where $G^-([a,b],X) = \{f \in G^-([a,b],X) \;/\; f^- = f\}$ is a closed subspace of $G([a,b],X)$, $f^-(t) = f(t_-), t \in]a,b]$, $f^-(a) = f(t_-)$, we had study optimization of linear functionals defined over a set of solutions of system $(K)+(F_\alpha)$. A characterization of regulated functions is given (see Theorem 3.1 of [4]) by:

Theorem 1. *Let $x : [a,b] \to X$ be a function. The following statements are equivalents:*

a) x is the uniform limit of a sequence of finite step functions;

b) $x \in G([a,b],X)$;

c) for every $\epsilon > 0$ there exists $d \in \mathbb{D}$ such that $\omega_d^{\cdot}(x) < \epsilon$.

It follows that the uniform limit of a sequence of regulated functions is a regulated function.

Lemma 1. *Let $f : [a,b]\times X \to X$ regulated as a function of the first variable and Lipschitz as a function of the second variable. Let $x \in G([a,b],X)$. Then $f_1 : s \in [a,b] \mapsto f(s,x(s)) \in X$ is a regulated function.*

Proof. If $t \in [a,b[$ we set $f(t,x(t_+)) \in X$. Then

$$\|f_1(s) - f(t,x(t_+))\| = \|f(s,x(s)) - f(t,x(t_+))\| \le c\,\|x(s) - x(t_+)\|\,.$$

Since $x \in G([a,b],X)$ we have $s \to t \implies x(s) \to x(t_+) \implies (s,x(s)) \to (t,x(t_+))$. So given $\epsilon/c > 0$ we have $\|f(s,x(s)) - f(t,x(t_+))\| < \epsilon$, provided $s \in [a,b]$, $0 < s-t < \delta$, that is, there exists $f_1(t_+) = f(t_+,x(t_+))$. By analogy we show that there exists $f_1(t_-) = f(t_-,x(t_-))$. □

In [3] we introduced, when $\phi \in BV([a,b],X^*)$, the notion of ϕ-convexity. Let $g : Y \to \overline{\mathbb{R}}$ be a convex function. We say that a function $f : [a,b]\times X \longrightarrow Y$ is convex with respect to g as a function of the second variable, or shortly, g-convex in the second variable, if $(g\circ f)_s$ is a convex function, $\forall s \in [a,b]$, where $(g\circ f)_s(x) = (g\circ f)(s,x)$ $\forall s \in [a,b]$ and $x \in X$. A particular case occurs when $Y = X$ and $g = \phi \in X^*$. For each ϕ fixed we will denote the set of all $f : [a,b]\times X \longrightarrow X$ that are regulated as a function of the first variable and ϕ-convex in the second variable by $G\cdot Conv_\phi([a,b]\times X, X)$. Note that the regularity of f in the first variable does not make influence in ϕ-convexity notion since this is valued by $(g \circ f)_s$. We introduce now a convex functional of Nemytskii type, $L_{\beta,f} : G([a,b],X) \to \overline{R}$, defined by

$$L_{\beta,f}[x] = \int_a^b \cdot \, d_s \, \beta(s) \cdot f(s, x(s))\,, \tag{2}$$

where $f \in G \cdot Conv_{\beta(s)}([a,b] \times X, X)$ and $\beta \in BV([a,b], X^*)$. Consider the optimal control convex problem

$$\min\{L_{\beta,f}(x_u)\ ,\ x_u \text{ satisties } (K + F_\alpha)\}\,. \tag{3}$$

2. Lower-Semicontinuity

We introduce now conditions on f and β such that $L_{\beta,f}$ will be a lower-semicontinuous function. Moreover, if f is a proper function then $L_{\beta,f}$ is too. As a consequence, Weierstrass theorem garantees (under compacity conditions) the existence of solution of (3).

We say that a function $f : [a,b] \times X \longrightarrow Y$ is lower semicontinuous with respect to $g : Y \to \mathbb{R}$ as a function of the second variable, or shortly, g-lower semicontinuous in the second variable, if $(g \circ f)_s : X \longrightarrow \mathbb{R}$ is a lower semicontinuous function, $\forall s \in [a,b]$, where $(g \circ f)_s(x) = (g \circ f)(s,x)$ $\forall s \in [a,b]$ and $x \in X$. We use this notion when $Y = X$ and $g = \phi \in X^*$. For each ϕ fixed we will denote the set of all $f : [a,b] \times X \longrightarrow X$ that are regulated as a function of the first variable and ϕ-lower-semicontinuous in the second variable by $G \cdot Lsc_\phi([a,b] \times X, X)$. We have that $f(s, x_n(s)) \to f(s, x(s))$, since that $x_n \to x$ in $G([a,b],X)$.

Theorem 2. *Suppose $f \in G \cdot Conv_{\beta(s)}([a,b] \times X, X)$, $\forall s \in [a,b]$ and $\beta \in BV([a,b], X^*)$. Then the functional $L_{\beta,f}$ is convex.*

Proof. Simple calculation. □

Theorem 3. *Suppose that $\beta \in BV([a,b], X^*)$ and $f \in G \cdot Lsc_{\beta(s)}([a,b] \times X, X)$, $\forall s \in [a,b]$. Then the functional $L_{\beta,f}$ is lower-semicontinuous.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of regulated functions with $y_n \in \Lambda(L_{\beta,f}, \lambda)$, for $n \in \mathbb{N}$, and such that $y_n \to y$, where for each $\lambda \in \mathbb{R}$,

$$\Lambda(L_{\beta,f}, \lambda) = \{x \in G([a,b],X)\ /\ L_{\beta,f}(x) \le \lambda\}\,,$$

are the level sets of the functional $L_{\beta,f}$, that is, for each $n \in \mathbb{N}$,

$$L_{\beta,f}[y_n] = \int_a^b \cdot \, \beta(s) \cdot f(s, y_n(s)) \le \lambda\,.$$

Since $\beta(s) \in X^*$ we have $\beta(s) \cdot f(s, y_n(s)) \leq \lambda$, $s \in [a, b]$, for each $\lambda \in \mathbb{R}$. So if $\beta(s) \in X^*$, and $\lambda \in \mathbb{R}$, it follows that $y_n \in \Lambda_{\beta(s)}(f, \lambda)$,

$$\Lambda_{\beta(s)}(f, \lambda) = \{x \in G([a, b], X)/ \ \beta(s) \cdot f(s, x(s)) \leq \lambda\}\,.$$

Since $f \in G \cdot Lsc_{\beta(s)}([a, b] \times X, X)$, $\forall s \in [a, b]$, we have that $\Lambda_{\beta(s)}(f, \lambda)$ are closed subsets in X. Then $y \in \Lambda_{\beta(s)}(f, \lambda)$, that is,

$$\beta(s) \cdot f(s, y(s)) \leq \lambda$$

and so $L_{\beta,f}(y) \leq \lambda$, that is, $y \in \Lambda(L_{\beta,f}, \lambda)$. Then $\Lambda(L_{\beta,f}, \lambda) \subset G([a, b], X)$ is a closed set and is lower-semicontinuous the functional $L_{\beta,f}$. In other words, is closed epigraph of $L_{\beta,f}$,

$$epi(L_{\beta,f}) = \{(x, \lambda) \in G([a, b], X) \times \mathbb{R}/ \ L_{\beta,f}(x) \leq \lambda\}$$

or, for all $x \in G([a, b], X)$ and a family of neighbourhood $\Upsilon(x)$ of x in $G([a, b], X)$,

$$L_{\beta,f}(x) = \lim_{y \to x} inf L_{\beta,f}(y) = \sup_{V \in \Upsilon(x)} \inf_{y \in V} L_{\beta,f}(y)\,. \qquad \square$$

As a consequence we have, by Weierstrass Theorem:

Theorem 4. *If $\Lambda \subset G([a, b], X)$ is a compact set, then $L_{\beta,f}$ has a minimun in K, that is, there exists solution of the optimization problem*

$$\min_{x \in \Lambda} L_{\beta,f}(x)\,.$$

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