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SHAPE DESIGN SENSITIVITY ANALYSIS (DSA) FOR 2-D STRUCTURAL FRACTURE USING EXTENDED FEM (XFEM)

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Abstract: This paper presents a novel design approach that facilitates design of structural components under fatigue and fracture for maximum service life. In this approach, a combination of extended finite element method and level sets method is employed for crack growth modeling and analysis of 2-D structures. This is followed by design sensitivity analysis, which evaluates sensitivity coefficients of crack growth rate and direction with respect to design variables. These coefficients help optimization algorithm to determine optimal geometric shape of the structure for maximum service life. A rectangular plate under shear loading is used to demonstrate feasibility of the proposed method.

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1. Introduction

One of the most technically challenging issue facing aerospace and mechani-

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cal engineers is the structural failure due to fatigue and fracture. Fatigue and fracture cause mechanical failures and safety hazards. Due to lack of adequate simulation tools for crack growth analysis, especially, for complex 3-dimensional structural components, heavy emphasis needs to be placed on physical testing, which is costly as well as time consuming. It is therefore necessary to incorporate accurate crack-growth simulation technique in design of components for maximum service life.

The finite element method is the most versatile and widely accepted method for crack propagation modeling and analysis. However, this method requires a very refined mesh to accurately capture the high stress gradients near crack tip region. Further, the structure needs to be re-meshed at the end of each crack growth cycle. These limitations make this method cumbersome for analysis of geometrically complex components. In contrast, the newly developed combination of eXtended Finite Element Method (XFEM) and Level Set Method (LSM), while operating within the familiar framework of FEM, models crack independent of the mesh; hence, same mesh can be re-used for crack propagation simulation [4]. It also uses special enrichment functions [7] to improve accuracy of the solution in the crack tip region, thereby allowing use of a relatively coarse mesh. Thus, it offers a major breakthrough for crack propagation simulation. The XFEM has been successfully demonstrated for 2-D as well as 3-D applications and has shown promising results [3] and [1] .

This research aims at further extending the method to support design of structural components by optimizing geometric shape for maximum service life for 2-D structures. This paper presents a design sensitivity analysis (DSA) method that serves as a fundamental step toward realizing such an objective. The shape DSA calculates gradients of crack propagation rate and propagation direction with respect to shape design variables, which are then supplied to a gradient-based optimization algorithm.

2. XFEM and LSM

XFEM is a computational technique in which special enrichment functions are used to incorporate the discontinuity caused by the crack surfaces and crack-tip fields into regular finite element approximation. The XFEM displacement approximation [7] for a vector valued function $u(x) : \Re^2 \rightarrow \Re^2$ is given as:

$$u^h(\mathbf{x}, t) = \sum_{i \in I} u_i(t) N_i(\mathbf{x}) + \sum_{j \in J} b_j(t) N_j(\mathbf{x}) H\left(\psi\left(\mathbf{x}, t\right)\right)$$

$$+\sum_{k\in K} N_k(\mathbf{x})\left(\sum_{l=1}^{4} a_k^l(t)B_l(r,\theta)\right), \quad (1)$$

where $N_i(\mathbf{x})$ is the shape function associated with the node i and t is a monotonically increasing time parameter that represents design cycles. J is the set of all nodes whose support is bisected by the crack (shown by circled nodes in Figure 1). The set K contains all nodes of the elements containing the crack tip (shown by squared nodes in Figure 1). In equation 1, the first term is the regular finite element approximation; the second term represents the Heaviside step function (H) employed to model the discontinuity due to crack, and the last term incorporates the near-tip asymptotic displacement fields using Branch functions, B_l, which are defined by

$$B_l(r,\theta) = \left\{\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right\}, \quad (2)$$

where (r,θ) are defined in a polar coordinate system at the crack tip and $\theta = 0$ is tangent to the crack. Because of the branch functions, a relatively coarse mesh can be used near the crack-tip region [7].

Level Set Method (LSM) is a numerical technique used to track motion of interfaces. In LSM, the crack is modeled using signed-distance functions ϕ and ψ (shown in Figure 2), which are stored at nodes. Thus, it is always possible to know which elements are cut by the crack and which elements contain the crack-tip. The LSM couples naturally with the XFEM and facilitates selection of nodes for enrichment. These enrichment functions appear in the form of extra degrees of freedom in the finite element stiffness matrix. At the end of a crack growth cycle, the signed-distance functions are updated to account for changes in crack geometry; hence, no re-meshing is required. Thus the XFEM and LSM provide an elegant scheme for crack growth simulation.

Domain form of interaction integral is used to compute stress intensity factors (SIF) [8] and [6]. This approach is theoretically more rigorous and is independent of the level of geometric complexity of the structure, thus making it especially attractive. Once the SIF's are obtained, a crack growth criterion is used to determine crack growth direction and the Paris law is used to determine crack growth rate [5].

3. Shape Design Sensitivity Analysis

The sensitivity coefficients allow optimization algorithm to decide which design parameters influence the performance functions to what extent. In this

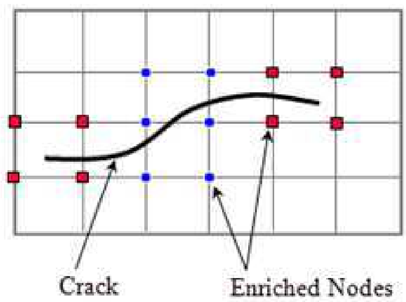

Figure 1. Nodal enrichment in XFEM

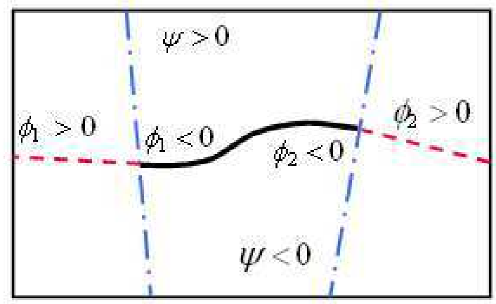

Figure 2. Crack representation using LSM

research, the focus is on computing sensitivity coefficients of the fracture parameters (crack growth rate and propagation angle) with respect to shape design variables. The crack propagation angle θ_c is determined using maximum hoop stress criterion [4] in this research. An expression for the same is given by

$$\theta_c = 2\tan^{-} 1\frac{1}{4}(K_I/K_{II} \pm \sqrt{(K_I/K_{II})^2+8})\,, \tag{3}$$

where K_I and K_{II} are the stress intensity factors for Mode I and Mode II fracture, respectively. Taking derivative of θ_c with respect to the shape design variables yields:

$$\begin{aligned}\frac{\partial\theta_c}{\partial b} &= 2\tan^{-1}\left[\frac{\partial K_I/\partial b}{4K_{II}} - \frac{K_I\cdot\partial K_I/\partial b}{4K_{II}^2} + \left(\frac{2K_I\cdot\partial K_I/\partial b}{K_{II}^2} - \frac{2K_I^2\cdot\partial K_{II}/\partial b}{K_{II}^2}\right)\right.\\ &\qquad \left.\times\left(2\sqrt{\frac{K_I^2}{K_{II}^2}+8}\right)^{-1}\right]\,. \end{aligned} \tag{4}$$

The crack growth rate is determined using Paris law, which is given by

$$\frac{da}{dN} = C\,(\Delta K)^m\,, \tag{5}$$

where da/dN is the crack growth per cycle, ΔK is the stress intensity factor range, and C and m are material parameters. For mixed-mode fracture problems, ΔK must be replaced by ΔK_{1eq}, which is the equivalent Mode I SIF [5]. Rearranging this equation and taking derivative with respect to shape design variables yields:

$$\frac{\partial\Delta N}{\partial b} = \frac{-m\cdot\Delta a}{C\,(\Delta K_{Ieq})^{m+1}}\cdot\frac{\partial\Delta K_{1eq}}{\partial b}\,. \tag{6}$$

As seen from equation (4) and equation (6), sensitivity coefficients of θ_c and ΔN depend on sensitivity coefficients of the SIFs. Hence the first step is to compute sensitivity coefficients of SIFs with respect to shape design variables. The SIFs are computed using domain form of the interaction integral [2]. For example,

$$K_I = \frac{E^{'}}{2} M^{(1,2)} \,, \tag{7}$$

where $M^{(1,2)}$ is the interaction integral and $E^{'}$ is the effective elastic modulus of the material for plane stress or plane strain state. $M^{(1,2)}$ can be written as

$$M^{(1,2)} = \int_A \left[\sigma_{ij}^{(1)} \frac{\partial z_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial z_i^{(1)}}{\partial x_1} - \sigma_{ij}^{(1)} \epsilon_{ij}^{(2)} \delta_{ij} \right] \frac{\partial q}{\partial x_j} dA \,. \tag{8}$$

Here σ_{ij} and z_i are components of stress tensor and displacement vector, respectively, A is a domain area inside an arbitrarily chosen counter-clockwise contour around the crack tip, and q is a weighing function whose value is zero along the boundary of the domain and unity at the crack tip. Superscript 1 denotes the actual mixed-mode state for the given problem, whereas superscript 2 denotes an auxiliary state corresponding to unit value of a SIF. To calculate K_I, an auxiliary state with $K_I = 1$ and $K_{II} = 0$ is chosen. To obtain sensitivity coefficients of the interaction integral with respect to shape design variables, equation (8) is differentiated with respect to design variable. Then the sensitivity coefficients of the SIFs can be easily obtained as follows:

$$\frac{\partial K_I}{\partial b} = \frac{E^{'}}{2} \frac{\partial M^{(1,I)}}{\partial b} \quad \text{and} \quad \frac{\partial K_{II}}{\partial b} = \frac{E^{'}}{2} \frac{\partial M^{(1,II)}}{\partial b} \,. \tag{9}$$

These can then be used to find sensitivity coefficients of θ_c and da/dN.

Presently, semi-analytical method is employed to compute sensitivity coefficients of fracture parameters. Consider the static equilibrium equation $K\mathbf{z} = F$, where K is the reduced stiffness matrix. Taking derivative with respect to design variable b, and rearranging terms,

$$\frac{\partial \mathbf{z}}{\partial b} = K^{-1} \left(\frac{\partial F}{\partial b} - \frac{\partial K}{\partial b} \mathbf{z} \right) \,. \tag{10}$$

Assuming that the load is independent of the design variable, the displacement material derivative can be easily obtained by solving equation if $\partial K/\partial b$ is known. $\partial K/\partial b$ is obtained using finite difference method. From displacement derivative, stress derivatives can be easily obtained, and therefore, material derivative of the interaction integral (from equation (8)) can be computed.

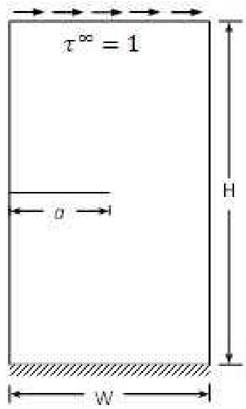

Figure 3. Edge crack under shear

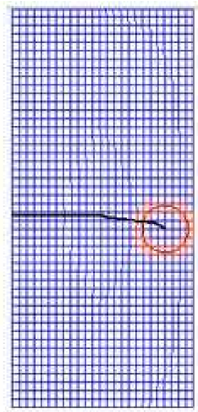

Figure 4. Crack growth simulation

Table 1. Crack Growth Simulation Results

Cycle	K_I ($ksi\sqrt{in}$)	K_{II} ($ksi\sqrt{in}$)	θ_c (Deg)
1	34.15	4.81	-15.45
2	38.13	0.58	1.73
3	41.97	6.30	-16.37
4	46.24	0.33	-0.83
5	50.09	4.29	-9.66
6	67.81	2.40	-4.04
7	78.78	5.09	-7.33
8	78.93	4.52	-6.52
9	66.84	14.72	-22.86
10	64.13	25.08	-34.86

Table 2. Comparison of results

	Reference	XFEM	% Diff.
K_I ($psi\sqrt{in}$)	34.13	34.15	-0.05
K_{II} ($psi\sqrt{in}$)	4.54	4.81	-5.94

Table 3. Displacement Sensitivity analysis results

No. of DOFs with less than 2% difference	602
No. of DOFs with greater than 5% difference	18

4. Numerical Example: Rectangular Plate with a Shear Load

Consider a rectangular plate with $W = 7$ in, $H = 16$ in, and an edge crack of $a = 3.5$ in as shown in Figure 3. The bottom face of the plate is fixed and a shear load of $\tau^{\infty} = 1$ acts on the top. The Young's modulus is $E = 30 \times 10^6$ and Poisson's ratio is $\nu = 0.25$. A 13*25 mesh is used that constitutes a total of 900 degrees of freedom, including the extra degrees of freedom incurred due to enriched nodes. A pre-determined crack growth increment (Δa), as discussed by Dolbow [6], is used. The crack growth simulation results are shown in Figure 4 and summarized in Tables 1 and 2.

Semi-analytical method (equation (10)) is used to compute sensitivity coefficients of displacement for the initial design. Sensitivity analysis results obtained by semi-analytical method are compared with those obtained from overall finite difference method. The difference between these results is measured for all degrees of freedom, and the results are summarized in Table 3.

5. Conclusion

This paper demonstrates semi-analytical method for computing design sensitivity coefficients of fracture parameters for 2-D structures. The SIF solutions obtained by XFEM agree very well with reference solution. Further, displacement sensitivity coefficients computed by semi-analytical method and overall finite difference method are in good agreement. Computation of $\partial K_I/\partial b$ and $\partial K_{II}/\partial b$ using these displacement sensitivity results is currently underway.

Also, the possibility of implementing a continuum-based material derivative technique for design sensitivity analysis is being investigated. In this technique, the governing equilibrium equations are differentiated prior to discretization, and hence are independent of the nature of approximation used. However, application of continuum approach for problems involving strong discontinuity requires further study. The final step is to incorporate an optimization algorithm, which will allow shape optimization structural components for maximum service life.

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