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ESTIMATING REPRODUCTIVE NUMBERS OF A CAMPUS DRINKING MODEL

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Abstract: Reproductive numbers are central to the epidemiological dynamics of any disease. However, estimating reproductive numbers have not been the explicit goal of college drinking researchers, since most of their research are not model driven. An epidemiological model capturing the dynamics of campus drinking is used to study how the "disease" of drinking is spread on campus. An optimization technique using known bounds for each parameter is used to estimate the reproductive numbers associated with campus drinking. A theorem establishing the conditions under which an endemic steady state exists is proposed and proved.

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1. Background

Alcohol abuse by college students has posed a significant public health concern for decades Ham et al [3], yet the use of mathematical models to analyze the dynamics of its spread is a relatively recent development Benedict [2]. In our previous work Manthey et al [4], we analyzed some of the dynamics of campus drinking. We have extended our previous analysis and applied an optimization procedure to estimate the reproductive numbers associated with campus drinking from bounds on the models parameters.

2. Mathematical Model

A college campus with the student population divided into three classes: non-drinkers (N), social drinkers (S), and problem drinkers (P) is considered. The student population is assumed to be constant.

As seen in Figure 1, students join campus in any one of the three drinking states and may transition from any drinking state to another. The "disease" of drinking is spread through social interaction and transitions to higher drinking states are modeled using the terms $\alpha NS, \kappa NP$, and γSP. Transitions to lower drinking states are assumed to be the result of a recovery process and are represented by $\beta S, \epsilon P$, and δP. It is assumed that there is a net positive flow towards a higher drinking state, an effect that has been termed the "college effect" Bachman et al [1].

The assumption of homogeneity is implicit in our model and implies that students within each class are similar with regards to their drinking behaviors and recruiting effectiveness. The dynamics of campus drinking are modeled by:

$$\frac{dN}{dt} = \eta - \eta N - \alpha NS - \kappa NP + \beta S + \epsilon P\,, \tag{1}$$

$$\frac{dS}{dt} = \sigma S - (\eta + \sigma)S + \alpha NS - \beta S - \gamma SP + \delta P\,, \tag{2}$$

$$\frac{dP}{dt} = \pi P - (\eta + \pi)P + \gamma SP + \kappa NP - \delta P - \epsilon P\,, \tag{3}$$

$$1 = N + S + P\,, \tag{4}$$

where N, S, and P have been rescaled as proportions. The model parameters are summarized in Table 1.

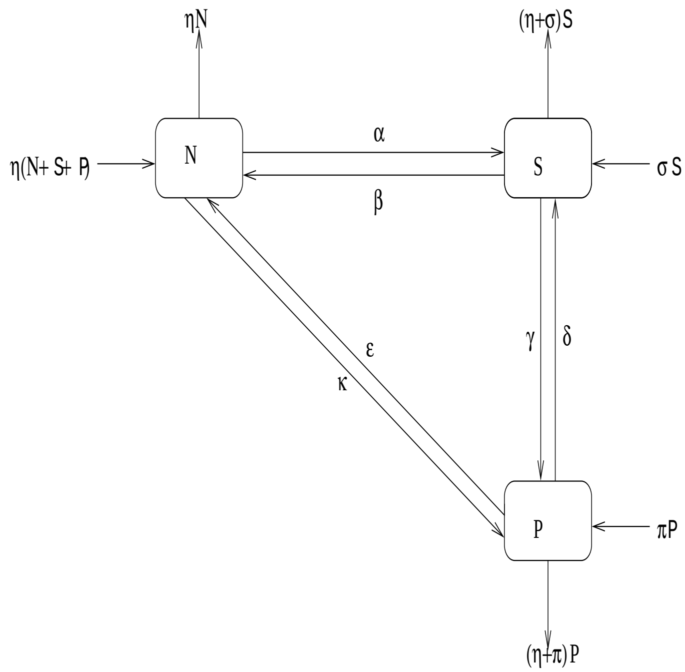

Figure 1: Schematic of the relationships between the three drinking classes on a college campus

Parameter	Description	Typical Value
α	transmission rate of N to S	1.0
β	recovery rate of S	0.2
γ	transmission rate of S to P	1.0
δ	recovery rate of P to S	0.2
ϵ	recovery rate of P to N	0.2
η	departure rate from campus	0.25
κ	transmission rate of N to P	1.5
σ	entrance rate of S	0.25
π	entrance rate of P	0.1

Table 1: Model parameters

3. Model Analysis

The equilibrium $(1, 0, 0)$ of system (1)-(4) represents a drinking free environment whose local stability may be determined from the eigenvalues of the Jacobian matrix

$$J(N,S,P) = \begin{bmatrix} -\eta-\alpha S-\kappa P & -\alpha N+\beta & -\kappa N+\epsilon \\ \alpha S & -\eta+\alpha N-\beta-\gamma P & -\gamma S+\delta \\ \kappa P & \gamma P & -\eta+\gamma S+\kappa N-\delta-\epsilon \end{bmatrix}.$$

Since there are two levels of infectives, there are two reproductive numbers based on the "disease-free" equilibrium. The first reproductive number R_0^S is defined as the average number of secondary cases generated by a typical social drinker in a non-drinking campus environment. The reproductive number R_0^S may be computed using $R_0^S = \lambda^* \times$ $(Infectious\ Period)$ $+1$, where λ^* is the dominant eigenvalue of $J(1,0,0)$. Thus

$$R_0^S = (\alpha-\eta-\beta)\times\left(\frac{1}{\eta+\beta}\right)+1 = \frac{\alpha}{\eta+\beta}. \tag{5}$$

The second reproductive number R_0^P is defined as the average number of secondary cases generated by a typical problem drinker in a non-drinking campus environment and is given by

$$R_0^P = \frac{\kappa}{\eta+\delta+\epsilon}. \tag{6}$$

The equilibrium $(1,0,0)$ is stable provided that $R_0^S < 1$ and $R_0^P < 1$.

A second equilibrium characterized by a lack of problem drinkers is given by $(\frac{\eta+\beta}{\alpha}, 1-\frac{\eta+\beta}{\alpha}, 0)$. If social drinking is not regarded as a problem, then we consider this as a second "disease-free" equilibrium. Expressing this equilibrium in terms of the reproductive number R_0^S yields $(\frac{1}{R_0^S}, 1-\frac{1}{R_0^S}, 0)$ from which $R_0^S > 1$. The third reproductive number, R_1 is defined as the average number of secondary cases generated by a typical problem drinker in a campus environment consisting of only non-drinkers and social drinkers. This reproductive number is computed using $J(\frac{\eta+\beta}{\alpha}, 1-\frac{\eta+\beta}{\alpha}, 0)$ and is given by

$$R_1 = \frac{\gamma(\alpha-\eta-\beta)+\kappa(\eta+\beta)}{\alpha(\eta+\delta+\epsilon)}. \tag{7}$$

It is possible for non-drinkers and social drinkers to coexist in equilibrium provided that $R_0^S > 1$ and $R_1 < 1$.

We now consider the conditions under which all three drinking states can coexist in equilibrium. From (1)-(4) we obtain

$$N = \frac{\eta-\gamma S+\delta+\epsilon}{\kappa}, \tag{8}$$

Reproductive No.	Max	Min	Model Estimate
R_0^S	27.00	0.63	2.22
R_0^P	25.00	0.59	2.31
R_1	34.30	0.71	10.06

Table 2: Estimates of reproductive numbers

$$P = \frac{\alpha\gamma S^2 + (\eta\kappa + \beta\kappa - \alpha\eta - \alpha\delta - \alpha\epsilon)S}{\kappa(\delta - \gamma S)}, \tag{9}$$

$$0 = S^2 + bS + c, \tag{10}$$

where $b = \frac{-\kappa\gamma+\alpha\delta+\epsilon\gamma+\eta\alpha-\delta\kappa+\alpha\epsilon-\beta\kappa+\eta\gamma+2\gamma\delta-\eta\kappa}{\kappa\gamma-\gamma^2-\alpha\gamma}$ and $c = \frac{-\delta\epsilon-\eta\delta-\delta^2+\delta\kappa}{\kappa\gamma-\gamma^2-\alpha\gamma}$. To obtain solutions that are relevant to the model situation, the model parameters must be chosen such that $0 < S < 1$. We note that the Jury conditions ensure that S lies in the unit disk $|S| < 1$. However this permits negative and even complex values for S. The following theorem provides the necessary and sufficient conditions under which S lies in the interval $0 < S < 1$.

Theorem. *(a) A necessary and sufficient condition that the equation* $S^2 + bS + c = 0$ *has two real roots in the interval* $(0, 1)$ *is that* $-2 < b < 0$ *and* $\max(0, -b-1) < c < \frac{1}{4}b^2$.

(b) A necessary and sufficient condition that the quadratic equation has one real root in the interval $(0, 1)$ *is that* $\min(0, -b-1) < c < \max(0, -b-1)$.

The theorem can easily be proved by considering the signs of the quadratic function at the endpoints. In addition to bounding S, an endemic equilibrium requires that N and P are also bounded. Using $0 < N < 1 - S$ and (9) we obtain the additional requirement $R_0^P \geq \max\left(1, \frac{\kappa}{\gamma}\right)$.

4. Estimates of Reproductive Numbers

We estimate the values of the reproductive numbers by solving the following nonlinear optimization problem using the conjugate gradient method.

Max/Min R_0^s, R_0^p, R_1, R_1^* subject to: $0.5 \leq \alpha \leq 3, 0.01 \leq \beta \leq 0.5, 0.5 \leq \gamma \leq 3, 0.01 \leq \delta \leq 0.5, 0.01 \leq \epsilon \leq 0.5, 0.1 \leq \eta \leq 3, 0.5 \leq \kappa \leq 3$, where R_0^s, R_0^p, R_1, R_1^* are given by equations (5, 6, 7). The estimates obtained are shown in Table 2.

5. Conclusion

We have established conditions under which an endemic equilibrium of the "disease" of drinking can exist on a college campus. In addition, we have estimated the reproductive numbers associated with campus drinking and are hopeful that this will lead to a better understanding of the dynamics of campus drinking and more effective intervention.

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