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ASYMPTOTIC ANALYSIS OF THE M/G/1 QUEUEING SYSTEM WITH ADDITIONAL OPTIONAL SERVICE AND NO WAITING CAPACITY

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Abstract: In this paper, we will do dynamic analysis for the M/G/1 queueing system with additional optional service and no waiting capacity by using functional analysis. First we will convert the mathematical model of the queueing system into an abstract Cauchy problem in a Banach space, next we will prove that the operator corresponding to the model generates a positive contraction C_0-semigroup, which is isometric for the initial value of the model. Thus we will obtain that the model has a unique positive time-dependent solution which satisfies probability condition. Third we will prove that the C_0-semigroup is a quasi-compact operator. From which we will deduce that the C_0-semigroup converges exponentially to a positive projection operator, and for special case, the time-dependent solution of the model converges strongly to the steady-state solution as time tends to infinite. Fourth we will discuss eigenvalues of the operator in the left half complex plane when the service rates are constants and then will give expression of the project operator by using the residue theorem in complex analysis. Last from the above steps we deduce that the time-dependent solution of the model converges exponentially to the steady-state solution of the model.

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1. Introduction

According to Madan [4], the M/G/1 queueing system with additional optional service and no waiting capacity can be described by the following system of equations:

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} + \lambda p_0(t) = q\int_0^\infty p_1(x,t)\mu_1(x)\mathrm{d}x + \int_0^\infty p_2(x,t)\mu_2(x)\mathrm{d}x, \tag{1}$$

$$\frac{\partial p_1(x,t)}{\partial x} + \frac{\partial p_1(x,t)}{\partial t} + \mu_1(x)p_1(x,t) = 0, \tag{2}$$

$$\frac{\partial p_2(x,t)}{\partial x} + \frac{\partial p_2(x,t)}{\partial t} + \mu_2(x)p_2(x,t) = 0, \tag{3}$$

$$p_1(0,t) = \lambda p_0(t), \quad t > 0, \tag{4}$$

$$p_2(0,t) = \overline{p}\int_0^\infty p_1(x,t)\mu_1(x)\mathrm{d}x, \quad t > 0, \tag{5}$$

$$p_0(0) = 1, \quad p_i(x,0) = 0, \quad i = 1,2. \tag{6}$$

Here $(x,t) \in [0,\infty) \times [0,\infty)$; $p_0(t)$ represents the probability that the server is idle at time t; $p_i(x,t)dx$ $(i = 1,2)$ represents the probability that at time t, the server is providing the i-th service with elapsed service time of the customer undergoing service lying between x and $x + \mathrm{d}x$; λ represent arrival rate of customers; $\overline{p}$ represents the probability that customers select second service after receiving the regular service; q represents the probability that customers leave the system after receiving the regular service; $\overline{p} + q = 1$; $\mu_i(x)\mathrm{d}x$ is the first order probability that the i-th service will be complete in time x and $x+\mathrm{d}x$ conditional that the same was not completed till time x.

In Madan [4], the author established the mathematical model of the M/G/1 queueing system with additional optional service and no waiting capacity by using supplementary variable technique and discussed the steady-state solution of the model. In Guo et al [1], the authors studied asymptotic property of the time-dependent solution of the model by discussing spectrum of the operator corresponding to the model and obtained that the time-dependent solution of the model converges strongly to the steady-state solution of the model. In this paper, first we will convert the model into an abstract Cauchy problem in a Banach space, next we will prove that the operator corresponding to the model generates a positive contraction C_0-semigroup, which is isometric for the initial value of the model, third we will prove that the model has a unique positive time-dependent solution which satisfies probability condition, fourth we will prove that the C_0-semigroup is quasi-compact, thus we will show that the C_0-semigroup converges exponentially a positive projection operator. In what follows, we will obtain that the time-dependent solution of the model converges strongly to the steady-state solution of the model, that is, the main result in Guo et al [1] is special case of our result. Fifth we will determine expression of the project operator when $\mu_i(x)$ $(i = 1, 2)$ are constants. Last we will prove that the time-dependent solution of the model converges exponentially to the steady-state solution of the model when $\mu_i(x)$ $(i = 1, 2)$ are constants.

We select state space X as follows:

$$X = \left\{p \in R \times L^1[0,\infty) \times L^1[0,\infty) \,\middle|\, \|p\| = |p_0| + \|p_1\|_{L^1[0,\infty)} + \|p_2\|_{L^1[0,\infty)} \right\}.$$

It is obvious that X is a Banach space. For simplicity, we introduce a notation as follows:

$$\Gamma = \begin{pmatrix} e^{-x} & 0 & 0 \\ \lambda e^{-x} & 0 & 0 \\ 0 & \overline{p}\mu_1(x) & 0 \end{pmatrix}.$$

In the following, define operators and their domains.

$$A\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}(x) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\frac{\mathrm{d}}{\mathrm{d}x} - \mu_1(x) & 0 \\ 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}x} - \mu_2(x) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \end{pmatrix},$$

$$D(A) = \left\{ p \in X \,\middle|\, \begin{array}{l} \frac{\mathrm{d}p_i(x)}{\mathrm{d}x} \in L^1[0,\infty), \quad p_i(x)\ (i = 1, 2) \text{ are absolutely} \\ \text{continuous functions and} \quad p(0) = \Gamma p(x) \end{array} \right\},$$

$$U\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}(x) = \begin{pmatrix} q\int_0^\infty p_1(x)\mu_1(x)\mathrm{d}x + \int_0^\infty p_2(x)\mu_2(x)\mathrm{d}x \\ 0 \\ 0 \end{pmatrix}, \quad D(U) = X.$$

Then the system of the above equations (1)-(6) can be rewritten as an abstract Cauchy problem in the Banach space X:

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = (A + U)p(t), \quad t \in [0, \infty), \tag{7}$$

$$p(0) = (1, 0, 0). \tag{8}$$

2. Main Results

Theorem 1. *If* $\mu_i(x)$ *satisfy* $M_i = \sup_{x\in[0,\infty)} \mu_i(x) < \infty, \quad i = 1, 2,$ *then* $A + U$ *generates a positive contraction* C_0*-semigroup* $T(t)$.

Proof. We split the proof into four steps. First we will prove that $(\gamma I - A)^{-1}$ exists and is bounded for $\gamma > M_1\overline{p}$, second we will prove that $D(A)$ is dense in X. Thus by using the Hille-Yosida Theorem we will obtain that A generates a C_0-semigroup $S(t)$, third we will verify that U is a bounded linear operator, therefore by using the above steps we will obtain that $A + U$ generates a C_0-semigroup $T(t)$. Finally, we will show that $A + U$ is a dispersive operator. Thus by applying the Phillips Theorem we will deduce the desired result.

For any given $y \in X$, consider equation $(\gamma I - A)p = y$. This is equivalent to

$$(\gamma + \lambda)p_0 = y_0, \tag{9}$$

$$\frac{\mathrm{d}p_1(x)}{\mathrm{d}x} = -(\gamma + \mu_1(x))p_1(x) + y_1(x), \tag{10}$$

$$\frac{\mathrm{d}p_2(x)}{\mathrm{d}x} = -(\gamma + \mu_2(x))p_2(x) + y_2(x), \tag{11}$$

$$p_1(0) = \lambda p_0, \tag{12}$$

$$p_2(0) = \overline{p} \int_0^\infty p_1(x)\mu_1(x)dx. \tag{13}$$

Solving $(9), (10)$ and (11) we have

$$p_0 = \frac{1}{\gamma + \lambda} y_0, \tag{14}$$

$$p_i(x) = a_i e^{-\gamma x - \int_0^x \mu_i(\tau)\mathrm{d}\tau} + e^{-\gamma x - \int_0^x \mu_i(\tau)\mathrm{d}\tau} \int_0^x y_i(\xi) e^{\gamma\xi + \int_0^\xi \mu_i(\tau)\mathrm{d}\tau} \mathrm{d}\xi,$$

$$i = 1, 2. \tag{15}$$

From (15) together with (12) and (13) we get

$$a_1 = p_1(0) = \lambda p_0. \tag{16}$$

$$a_2 = p_2(0) = \overline{p} \int_0^\infty p_1(x)\mu_1(x)\mathrm{d}x. \tag{17}$$

By using (14), (15), (16), (17) and the Fubini Theorem we deduce (without loss of generality assume that $\gamma > 0$)

$$\|p_1\|_{L^1[0,\infty)} \leq \frac{\lambda}{\gamma + \lambda} |y_0| \int_0^\infty e^{-\gamma x - \int_0^x \mu_1(\tau)\mathrm{d}\tau} \mathrm{d}x$$

$$
\begin{aligned}
&+\int_0^\infty e^{-\gamma x-\int_0^x \mu_1(\tau)\mathrm{d}\tau}\int_0^x |y_1(\xi)|e^{\gamma\xi+\int_0^\xi \mu_1(\tau)\mathrm{d}\tau}\mathrm{d}\xi\mathrm{d}x\\
&\le \frac{\lambda}{\gamma+\lambda}|y_0|\int_0^\infty e^{-\gamma x}\mathrm{d}x+\int_0^\infty e^{-\gamma x}\int_0^x |y_1(\xi)|e^{\gamma\xi}e^{\int_0^\xi \mu_1(\tau)\mathrm{d}\tau-\int_0^x \mu_1(\tau)\mathrm{d}\tau}\mathrm{d}\xi\mathrm{d}x\\
&\le \frac{\lambda}{\gamma(\gamma+\lambda)}|y_0|+\int_0^\infty e^{-\gamma x}\int_0^x |y_1(\xi)|e^{\gamma\xi}e^{-\int_\xi^x \mu_1(\tau)\mathrm{d}\tau}\mathrm{d}\xi\mathrm{d}x\\
&\le \frac{\lambda}{\gamma(\gamma+\lambda)}|y_0|+\int_0^\infty e^{-\gamma x}\int_0^x |y_1(\xi)|e^{\gamma\xi}\mathrm{d}\xi\mathrm{d}x\\
&\le \frac{\lambda}{\gamma(\gamma+\lambda)}|y_0|+\int_0^\infty |y_1(\xi)|e^{\gamma\xi}\int_\xi^\infty e^{-\gamma x}\mathrm{d}x\mathrm{d}\xi\\
&\le \frac{\lambda}{\gamma(\gamma+\lambda)}|y_0|+\frac{1}{\gamma}\|y_1\|_{L^1[0,\infty)}. \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \|p_2\|_{L^1[0,\infty)} &\le \int_0^\infty\left(\overline{p}\int_0^\infty |p_1(x)|\mu_1(x)\mathrm{d}x\right)e^{-\gamma x-\int_0^x \mu_2(\tau)\mathrm{d}\tau}\mathrm{d}x\\
&+\int_0^\infty e^{-\gamma x-\int_0^x \mu_2(\tau)\mathrm{d}\tau}\int_0^x |y_2(\xi)|e^{\gamma\xi+\int_0^\xi \mu_2(\tau)\mathrm{d}\tau}\mathrm{d}\xi\mathrm{d}x\\
&\le M_1\overline{p}\|p_1\|_{L^1[0,\infty)}\int_0^\infty e^{-\gamma x}\mathrm{d}x+\frac{1}{\gamma}\|y_2\|_{L^1[0,\infty)}\\
&\le \frac{M_1\overline{p}}{\gamma}\left[\frac{\lambda}{\gamma(\gamma+\lambda)}|y_0|+\frac{1}{\gamma}\|y_1\|_{L^1[0,\infty)}\right]+\frac{1}{\gamma}\|y_2\|_{L^1[0,\infty)}\\
&= \frac{M_1\overline{p}\lambda}{\gamma^2(\gamma+\lambda)}|y_0|+\frac{M_1\overline{p}}{\gamma^2}\|y_1\|_{L^1[0,\infty)}+\frac{1}{\gamma}\|y_2\|_{L^1[0,\infty)}. \qquad (19)
\end{aligned}
$$

In (18) and (19) we used the following inequalities:

$$e^{-\int_0^x \mu_i(\tau)\mathrm{d}\tau}\le 1, \quad \text{for} \quad x\in[0,\infty); \quad e^{-\int_\xi^x \mu_i(\tau)\mathrm{d}\tau}\le 1, \quad \text{for} \quad x\ge\xi\ge 0.$$

By (14), (18) and (19) it follows that (without loss of generality assume that $\gamma > M_1\overline{p}$)

$$
\begin{aligned}
\|p\| &= |p_0|+\|p_1\|_{L^1[0,\infty)}+\|p_2\|_{L^1[0,\infty)} = \frac{1}{\gamma+\lambda}|y_0|+\frac{\lambda}{\gamma(\gamma+\lambda)}|y_0|+\frac{1}{\gamma}\|y_1\|_{L^1[0,\infty)}\\
&+\frac{M_1\overline{p}\lambda}{\gamma^2(\gamma+\lambda)}|y_0|+\frac{M_1\overline{p}}{\gamma^2}\|y_1\|_{L^1[0,\infty)}+\frac{1}{\gamma}\|y_2\|_{L^1[0,\infty)}\\
&= \frac{\gamma^2+\gamma\lambda+M_1\overline{p}\lambda}{\gamma^2(\gamma+\lambda)}|y_0|+\frac{\gamma+M_1\overline{p}}{\gamma^2}\|y_1\|_{L^1[0,\infty)}+\frac{1}{\gamma}\|y_2\|_{L^1[0,\infty)}\\
&\le \frac{1}{\gamma-M_1\overline{p}}\left(|y_0|+\|y_1\|_{L^1[0,\infty)}+\|y_2\|_{L^1[0,\infty)}\right)=\frac{1}{\gamma-M_1\overline{p}}\|y\|. \qquad (20)
\end{aligned}
$$

(20) shows that, when $\gamma > M_1\overline{p}$,

$$(\gamma I - A)^{-1} : X \to D(A), \quad \|(\gamma I - A)^{-1}\| \leq \frac{1}{\gamma - M_1\overline{p}}.$$

Turning to the second step, set

$$L = \{p(x) = (p_0, p_1(x), p_2(x))|\ p_0 \in R, p_i(x) \in C_0^{\infty}[0,\infty), i = 1,2\},$$

then by Gupur et al [3] it is obvious that L is dense in X. If we define

$$V = \left\{ p(x) = (p_0, p_1(x), p_2(x)) \;\middle|\; \begin{array}{l} p_i(x) \in C_0^{\infty}[0,\infty), \quad \exists \alpha_i \text{ such that} \\ p_i(x) = 0, \text{ for } x \in [0, \alpha_i], i = 1,2 \end{array} \right\},$$

then V is dense in L. Therefore, in order to prove that $D(A)$ is dense in X, it suffices to prove that $D(A)$ is dense in V. Taking any $p(x) = (p_0, p_1(x), p_2(x)) \in V$, then there exist positive numbers α_i such that $p_i(x) = 0$, for all $x \in [0, \alpha_i]$ $(i = 1, 2)$. Which result $p_i(x) = 0$, for $x \in [0, 2s]$, here $0 < 2s = \min\{\alpha_1, \alpha_2\}$. Define

$$f^s(0) = (f_0^s, f_1^s(0), f_2^s(0)) = (p_0, \lambda p_0, \overline{p}\int_{2s}^{\infty} p_1(x)\mu_1(x)\mathrm{d}x),$$
$$f^s(x) = (f_0^s, f_1^s(x), f_2^s(x)),$$

where

$$f_1^s(x) = \begin{cases} f_1^s(0)\left(1 - \frac{x}{s}\right)^2 & \text{if } x \in [0, s), \\ -c_1(x-s)^2(x-2s)^2 & \text{if } x \in [s, 2s), \\ p_1(x) & \text{if } x \in [2s, \infty), \end{cases}$$

$$f_2^s(x) = p_2(x), \quad c_1 = \frac{\int_0^s f_1^s(0)\left(1 - \frac{x}{s}\right)^2 \mu_1(x)dx}{\int_s^{2s}(x-s)^2(x-2s)^2\mu_1(x)dx}.$$

It is easy to verify that $f^s \in D(A)$. Moreover

$$\|p - f^s\| = \int_0^s |f_1^s(0)| \left(1 - \frac{x}{s}\right)^2 \mathrm{d}x + |c_1| \int_s^{2s} (x-s)^2(x-2s)^2 dx$$
$$= |f_1^s(0)|\frac{s}{3} + |c_1|\frac{s^5}{30} \to 0, \quad \text{as} \quad s \to 0.$$

Which shows that $D(A)$ is dense in V. In other words, $D(A)$ is dense in X.

From the first step, second step and Hille-Yosida Theorem, we know that A generates a C_0-semigroup $S(t)$. For any $p \in X$, from the definition of U we have

$$\|Up\| \leq q\int_0^{\infty} |p_1(x)|\mu_1(x)\mathrm{d}x + \int_0^{\infty} |p_2(x)|\mu_2(x)\mathrm{d}x$$
$$\leq qM_1 \int_0^{\infty} |p_1(x)|\mathrm{d}x + M_2 \int_0^{\infty} |p_2(x)|\mathrm{d}x \leq \max\{qM_1, M_2\}\|p\|.$$

This shows that U is a bounded operator. It is easy to see that U is linear. Hence the perturbation theorem of C_0-semigroup, we obtain that $(A+U)$ generates a C_0-semigroup $T(t)$.

As far as the third step concerned, by using Philips Theorem we will prove that $T(t)$ is a positive contraction operator.

For $p \in D(A)$, we may choose

$$\phi(x) = \left(\frac{[p_0]^+}{p_0}, \frac{[p_1(x)]^+}{p_1(x)}, \frac{[p_2(x)]^+}{p_2(x)} \right),$$

here

$$[p_0]^+ = \begin{cases} p_0 & \text{if} \quad p_0 > 0, \\ 0 & \text{if} \quad p_0 \le 0, \end{cases} \qquad [p_i(x)]^+ = \begin{cases} p_i(x) & \text{if} \quad p_i(x) > 0, \\ 0 & \text{if} \quad p_i(x) \le 0, \end{cases} \qquad i = 1, 2.$$

If we define $W_i = \{x \in [0, \infty) | p_i(x) > 0\}$ and $Q_i = \{x \in [0, \infty) | p_i(x) \le 0\}$ $\quad (i = 1, 2)$, then we have

$$\begin{aligned} \int_0^\infty \frac{\mathrm{d}p_i(x)}{\mathrm{d}x} \frac{[p_i(x)]^+}{p_i(x)} \mathrm{d}x &= \int_{W_i} \frac{\mathrm{d}p_i(x)}{\mathrm{d}x} \frac{[p_i(x)]^+}{p_i(x)} \mathrm{d}x + \int_{Q_i} \frac{\mathrm{d}p_i(x)}{\mathrm{d}x} \frac{[p_i(x)]^+}{p_i(x)} \mathrm{d}x \\ &= \int_{W_i} \frac{\mathrm{d}p_i(x)}{\mathrm{d}x} \frac{[p_i(x)]^+}{p_i(x)} \mathrm{d}x = \int_{W_i} \frac{\mathrm{d}p_i(x)}{\mathrm{d}x} dx \\ &= \int_0^\infty \frac{[p_i(x)]^+}{p_i(x)} \mathrm{d}x = -[p_i(0)]^+, \quad i = 1, 2. \end{aligned} \tag{21}$$

$$\int_0^\infty p_i(x)\mu_i(x)\mathrm{d}x \le \int_0^\infty [p_i(x)]^+ \mu_i(x)\mathrm{d}x, \quad i = 1, 2. \tag{22}$$

By (21), (22) and the boundary conditions on $p \in D(A)$ we obtain

$$\begin{aligned} \langle (A+U)p, \phi \rangle = & \frac{[p_0]^+}{p_0} \left\{ -\lambda p_0 + q \int_0^\infty p_1(x)\mu_1(x)\mathrm{d}x + \int_0^\infty p_2(x)\mu_2(x)\mathrm{d}x \right\} \\ & + \int_0^\infty \frac{[p_1(x)]^+}{p_1(x)} \left\{ -\frac{\mathrm{d}p_1(x)}{\mathrm{d}x} - \mu_1(x)p_1(x) \right\} \mathrm{d}x \\ & + \int_0^\infty \frac{[p_2(x)]^+}{p_2(x)} \left\{ -\frac{\mathrm{d}p_2(x)}{\mathrm{d}x} - \mu_2(x)p_2(x) \right\} \mathrm{d}x \\ \le & -\lambda [p_0]^+ + \frac{[p_0]^+}{p_0} q \int_0^\infty [p_1(x)]^+ \mu_1(x)\mathrm{d}x + \frac{[p_0]^+}{p_0} \int_0^\infty [p_2(x)]^+ \mu_2(x)\mathrm{d}x \\ & + [p_1(0)]^+ - \int_0^\infty [p_1(x)]^+ \mu_1(x)\mathrm{d}x + [p_2(0)]^+ - \int_0^\infty [p_2(x)]^+ \mu_2(x)\mathrm{d}x \\ \le & -\lambda [p_0]^+ + \frac{[p_0]^+}{p_0} q \int_0^\infty [p_1(x)]^+ \mu_1(x)\mathrm{d}x + \frac{[p_0]^+}{p_0} \int_0^\infty [p_2(x)]^+ \mu_2(x)\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\lambda[p_0]^+ - \int_0^\infty [p_1(x)]^+\mu_1(x)\mathrm{d}x \\
&+\overline{p}\int_0^\infty [p_1(x)]^+\mu_1(x)\mathrm{d}x - \int_0^\infty [p_2(x)]^+\mu_2(x)\mathrm{d}x \\
&\le \left(\frac{[p_0]^+}{p_0}q - 1 + \overline{p}\right)\int_0^\infty [p_1(x)]^+\mu_1(x)\mathrm{d}x + \left(\frac{[p_0]^+}{p_0} - 1\right)\int_0^\infty [p_2(x)]^+\mu_2(x)\mathrm{d}x \\
&\le (q+\overline{p}-1)\int_0^\infty [p_1(x)]^+\mu_1(x)\mathrm{d}x + \left(\frac{[p_0]^+}{p_0} - 1\right)\int_0^\infty [p_2(x)]^+\mu_2(x)\mathrm{d}x \\
&= \left(\frac{[p_0]^+}{p_0} - 1\right)\int_0^\infty [p_2(x)]^+\mu_2(x)\mathrm{d}x \le 0. \quad (23)
\end{aligned}
$$

(23) shows that $A+U$ is a dispersive operator. Hence from the above steps and Phillips theorem we deduce that $A+U$ generates a positive contraction C_0-semigroup (see Gupur et al [3]). By the uniqueness of C_0-semigroup, we know that this positive contraction C_0-semigroup is $T(t)$. Thus we complete the proof of Theorem 1. □

Remark 1. From the estimating processes of (23) it is easy to see

$$\langle Ap, \phi\rangle = (\overline{p}-1)\int_0^\infty [p_1(x)]^+\mu_1(x)dx + \left(\frac{[p_0]^+}{p_0} - 1\right)\int_0^\infty [p_2(x)]^+\mu_2(x)dx \le 0.$$

Thus by using the Phillips Theorem we deduce that $S(t)$ is a positive contraction C_0-semigroup.

It is easy to verify that X^*, dual space of X, is

$$X^* = \big\{q^* \in R\times L^\infty[0,\infty)\times L^1[0,\infty) \mid |||q^*||| = \sup\{|q_0^*|, \|q_1^*\|_{L^1[0,\infty)}, \|q_2^*\|_{L^1[0,\infty)}\}\big\}.$$

It is clear that X^* is a Banach space. In X, take

$$Y = \{p\in X \mid p_0\ge 0, \quad p_1(x)\ge 0, \quad p_2(x)\ge 0, \quad \forall x\in[0,\infty)\},$$

then Y is a cone in X. For $p\in D(A)\cap Y$ we take $q^*(x) = \|p\|(1,1,1)$, then it is obvious that $q^*\in X^*$ and noting $\overline{p}+q=1$,

$$
\begin{aligned}
&\langle (A+U)p, q^*\rangle = \|p\|\left\{-\lambda p_0 + q\int_0^\infty p_1(x)\mu_1(x)dx + \int_0^\infty p_2(x)\mu_2(x)dx\right\} \\
&+\int_0^\infty \|p\|\left\{-\frac{dp_1(x)}{dx} - \mu_1(x)p_1(x)\right\}dx + \int_0^\infty \|p\|\left\{-\frac{dp_2(x)}{dx} - \mu_2(x)p_2(x)\right\}dx \\
&= -\lambda p_0\|p\| + q\|p\|\int_0^\infty p_1(x)\mu_1(x)dx \\
&+\|p\|\int_0^\infty p_2(x)\mu_2(x)dx + \|p\|p_1(0) - \|p\|\int_0^\infty p_1(x)\mu_1(x)dx
\end{aligned}
$$

$$+\|p\|p_2(0)-\|p\|\int_0^\infty p_2(x)\mu_2(x)dx$$
$$=-\lambda p_0\|p\|+q\|p\|\int_0^\infty p_1(x)\mu_x(x)dx+\|p\|\int_0^\infty p_2(x)\mu_2(x)dx$$
$$+\lambda p_0\|p\|-\|p\|\int_0^\infty p_1(x)\mu_1(x)dx+\|p\|\overline{p}\int_0^\infty p_1(x)\mu_1(x)dx$$
$$-\|p\|\int_0^\infty p_2(x)\mu_2(x)dx$$
$$=(q+\overline{p})\|p\|\int_0^\infty p_1(x)\mu_1(x)dx-\|p\|\int_0^\infty p_1(x)\mu_1(x)dx=0. \quad (1)$$

This shows that $A+U$ is conservative for

$$\theta(p)=\{q^*\in X^* \mid \langle p,q^*\rangle=\|p\|^2=\||q^*\||^2\}.$$

Thus, since $p(0)\in D(A^2)\cap Y$, by using the Fattorini Theorem we deduce the following result (see Gupur et al [3]).

Theorem 2. *$T(t)$ is isometric for the initial value $p(0)$, that is, $\|T(t)p(0)\|=\|p(0)\|, \quad \forall t\in[0,\infty)$.*

By combining Theorem 1 with Theorem 2 we obtain well-posedness of the system (7)-(8).

Theorem 3. *The system (7)-(8) has a unique positive time-dependent solution $p(x,t)$ that satisfies $\|p(\cdot,t)\|=1, \quad \forall t\in[0,\infty)$.*

Proof. From Theorem 1 and Gupur et al [3] we know that the system (7)-(8) has a unique positive time-dependent solution $p(x,t)$, which can be expressed as

$$p(x,t)=T(t)p(0), \quad \forall t\in[0,\infty).$$

From this together with Theorem 2 we derive

$$\|p(\cdot,t)\|=\|T(t)p(0)\|=\|p(0)\|=1, \quad \forall t\in[0,\infty).$$

This just reflects the physical meaning of $p(x,t)$. The proof of this theorem is completed. □

In the following, we will study asymptotic property of the above $p(x,t)$.

Lemma 1. *If $p(x,t)=S(t)\phi(x)$ is the solution of the following system*

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t}=Ap(t),\ \forall t\in[0,\infty), \quad (24)$$
$$p(0)=\phi(x), \quad (25)$$

then

$$p(x,t)=\begin{cases}\begin{pmatrix}\phi_0 e^{-\lambda t}\\ p_1(0,t-x)e^{-\int_0^x \mu_1(\tau)\mathrm{d}\tau}\\ p_2(0,t-x)e^{-\int_0^x \mu_2(\tau)\mathrm{d}\tau}\end{pmatrix} & \textit{if} \quad x<t,\\ \begin{pmatrix}\phi_0 e^{-\lambda t}\\ \phi_1(x-t)e^{-\int_{x-t}^x \mu_1(\tau)\mathrm{d}\tau}\\ \phi_2(x-t)e^{-\int_{x-t}^x \mu_2(\tau)\mathrm{d}\tau}\end{pmatrix} & \textit{if} \quad x>t.\end{cases}$$

$p_1(0,t-x)$ *and* $p_2(0,t-x)$ *are given by (4) and (5) respectively.*

Proof. Since $p(x,t)=S(t)\phi(x)$ is the solution of the system (24)-(25), $p(x,t)$ satisfies

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t}=-\lambda p_0(t), \tag{26}$$

$$\frac{\partial p_1(x,t)}{\partial x}+\frac{\partial p_1(x,t)}{\partial t}=-\mu_1(x)p_1(x,t), \tag{27}$$

$$\frac{\partial p_2(x,t)}{\partial x}+\frac{\partial p_2(x,t)}{\partial t}=-\mu_2(x)p_2(x,t), \tag{28}$$

$$p_1(0,t)=\lambda p_0(t), \tag{29}$$

$$p_2(0,t)=\overline{p}\int_0^\infty p_1(x,t)\mu_1(x)dx, \tag{30}$$

$$p_0(0)=\phi_0, \quad p_i(x,0)=\phi_i(x), \quad i=1,2. \tag{31}$$

If we define $\xi=x-t$ and $Q_i(t)=p_i(\xi+t,t)$ $(i=1,2)$, then from (27) and (28) we deduce

$$\frac{\mathrm{d}Q_i(t)}{\mathrm{d}t}=-\mu_i(\xi+t)Q_i(t), \quad i=1,2. \tag{32}$$

If $\xi<0$ (i.e., $x<t$), then by integrating (32) from $-\xi$ to t and using $Q_i(-\xi)=p_i(0,-\xi)=p_i(0,t-x), i=1,2,$ and by using new integral variable $y=\xi+\tau$ we obtain

$$\begin{aligned}p_i(x,t)&=Q_i(t)=Q_i(-\xi)e^{-\int_{-\xi}^t \mu_i(\xi+\tau)\mathrm{d}\tau}\\ &=p_i(0,t-x)e^{-\int_0^{\xi+t}\mu_i(y)\mathrm{d}y}=p_i(0,t-x)e^{-\int_0^x \mu_i(\tau)\mathrm{d}\tau}, \quad i=1,2.\end{aligned} \tag{33}$$

By solving (26) we have

$$p_0(t)=\phi_o e^{-\lambda t}. \tag{34}$$

If $\xi>0$ (i.e., $x>t$), then by integrating (32) from 0 to t and using $Q_i(0)=p_i(\xi,0)=\phi_i(\xi)=\phi_i(x-t), i=1,2,$ we derive

$$p_i(x,t)=Q_i(t)=Q_i(0)e^{-\int_0^t \mu_i(\xi+\tau)\mathrm{d}\tau}$$

$$= \phi_i(x-t)e^{-\int_\xi^{\xi+t}\mu_i(y)\mathrm{d}y} = \phi_i(x-t)e^{-\int_{x-t}^{x}\mu_i(\tau)\mathrm{d}\tau}, \quad i=1,2. \quad (35)$$

By combining (33) and (34) with (35) we know that the result of Lemma 1 is right. □

For $\phi \in X$, we define two operators as follows

$$(W(t)\phi)(x) = \begin{cases} 0 & \text{if} \quad x \in [0,t), \\ (S(t)\phi)(x) & \text{if} \quad x \in [t,\infty), \end{cases}$$

$$(Z(t)\phi)(x) = \begin{cases} (S(t)\phi)(x) & \text{if} \quad x \in [0,t), \\ 0 & \text{if} \quad x \in [t,\infty), \end{cases}$$

then $S(t)\phi = W(t)\phi + Z(t)\phi, \quad \forall \phi \in X$.

From Gupur [2] we know the following result:

Proposition 1. *A bounded subset Y of X is relatively compact if and only if the following two conditions hold simultaneously.*

(1) $\sum_{i=1}^{2} \lim_{h\to 0} \int_0^\infty |\phi_i(x+h)-\phi_i(x)|\mathrm{d}x = 0$, *uniformly for* $\phi = (\phi_0,\phi_1,\phi_2) \in Y$.

(2) $\sum_{i=1}^{2} \lim_{h\to \infty} \int_h^\infty |\phi_i(x)|\mathrm{d}x = 0$, *uniformly for* $\phi = (\phi_0,\phi_1,\phi_2) \in Y$.

Theorem 4. *If $\mu_i(x)$ $(i=1,2)$ are Lipschitz continuous and satisfy $0 < \underline{\mu} \le \mu_i(x) \le \overline{\mu} < \infty$, then $Z(t)$ is a compact operator in X.*

Proof. From the definition of $Z(t)$ and Proposition 1, we know that it is enough to prove the condition (1) in Proposition 1. By Lemma 1 we have

$$\begin{aligned}
&\sum_{i=1}^{2}\int_0^t |p_i(x+h,t)-p_i(x,t)|\mathrm{d}x \\
&\quad = \sum_{i=1}^{2}\int_0^t \left| p_i(0,t-x-h)e^{-\int_0^{x+h}\mu_i(\tau)\mathrm{d}\tau} - p_i(0,t-x)e^{-\int_0^{x}\mu_i(\tau)d\tau}\right| \mathrm{d}x \\
&\quad = \sum_{i=1}^{2}\int_0^t \Big| p_i(0,t-x-h)e^{-\int_0^{x+h}\mu_i(\tau)\mathrm{d}\tau} - p_i(0,t-x-h)e^{-\int_0^{x}\mu_i(\tau)\mathrm{d}\tau} \\
&\qquad\quad + p_i(0,t-x-h)e^{-\int_0^{x}\mu_i(\tau)\mathrm{d}\tau} - p_i(0,t-x)e^{-\int_0^{x}\mu_i(\tau)\mathrm{d}\tau}\Big| \mathrm{d}x \\
&\qquad \le \sum_{i=1}^{2}\int_0^t |p_i(0,t-x-h)| \left| e^{-\int_0^{x+h}\mu_i(\tau)\mathrm{d}\tau} - e^{-\int_0^{x}\mu_i(\tau)\mathrm{d}\tau}\right| \mathrm{d}x \\
&\qquad\qquad + \sum_{i=1}^{2}\int_0^t |p_i(0,t-x-h)-p_i(0,t-x)|e^{-\int_0^{x}\mu_i(\tau)\mathrm{d}\tau}dx. \quad (36)
\end{aligned}$$

In the following, we will estimate the first term in (36). By noting (29) and using Lemma 1, for $t-x-h \geq 0$, we have

$$|p_1(0,t-x-h)| = |\lambda p_0(t-x-h)| = \lambda|\phi_0|e^{-\lambda(t-x-h)} \leq \lambda|\phi_0| \leq \lambda\|\phi\|_X, \quad (37)$$

$$\begin{aligned}|p_2(0,t-x-h)| &= \left|\overline{p}\int_0^\infty p_1(y,t-x-h)\mu_1(y)\mathrm{d}y\right| \\ &\leq \overline{p\mu}\int_0^\infty |p_1(y,t-x-h)|dy = \overline{p\mu}\|p_1(\cdot,t-x-h)\|_{L^1[0,\infty)} \\ &\leq \overline{p\mu}\|S(t)\phi\|_X \leq \overline{p\mu}\|\phi\|_X. \quad (38)\end{aligned}$$

$$\begin{aligned}&\Longrightarrow \sum_{i=1}^{2}\int_0^t |p_i(0,t-x-h)|\left|e^{-\int_0^{x+h}\mu_i(\tau)\mathrm{d}\tau} - e^{-\int_0^{x}\mu_i(\tau)\mathrm{d}\tau}\right|\mathrm{d}x \\ &\leq \max\{\lambda,\overline{p\mu}\}\|\phi\|_X \sum_{i=1}^{2}\int_0^t \left|e^{-\int_0^{x+h}\mu_i(\tau)\mathrm{d}\tau} - e^{-\int_0^{x}\mu_i(\tau)\mathrm{d}\tau}\right|\mathrm{d}x \to 0, \\ &\qquad \text{as} \quad h\to 0, \text{ uniformly for } \quad \phi. \quad (39)\end{aligned}$$

In the following, we will estimate the second term in (36). By using Lemma 1, (29), (30) and Lipschitz continuity (without loss of generality assume that Lipschitz constant is 1) we calculate

$$\begin{aligned}&|p_1(0,t-x-h) - p_1(0,t-x)| = |\lambda p_0(t-x-h) - \lambda p_0(t-x)| \\ &= \lambda|p_0(t-x-h) - p_0(t-x)| = \lambda\left|\phi_0 e^{-\lambda(t-x-h)} - \phi_0 e^{-\lambda(t-x)}\right| \\ &\leq \lambda|\phi_0|\left|e^{-\lambda(t-x-h)} - e^{-\lambda(t-x)}\right| \leq \lambda\left|e^{-\lambda(t-x-h)} - e^{-\lambda(t-x)}\right|\|\phi\|_X \to 0, \\ &\qquad \text{as} \quad h\to 0, \text{uniformly for} \quad \phi. \quad (40)\end{aligned}$$

$$\begin{aligned}&|p_2(0,t-x-h) - p_2(0,t-x)| \\ &= \left|\overline{p}\int_0^\infty p_1(y,t-x-h)\mu_1(y)\mathrm{d}y - \overline{p}\int_0^\infty p_1(y,t-x)\mu_1(y)\mathrm{d}y\right| \\ &= \overline{p}\left|\int_0^{t-x-h} p_1(y,t-x-h)\mu_1(y)dy + \int_{t-x-h}^\infty p_1(y,t-x-h)\mu_1(y)dy\right. \\ &\quad \left. - \int_0^{t-x} p_1(y,t-x)\mu_1(y)dy - \int_{t-x}^\infty p_1(y,t-x)\mu_1(y)dy\right| \\ &= \overline{p}\left|\int_0^{t-x-h} p_1(0,t-x-h-y)e^{-\int_0^y \mu_1(\tau)d\tau}\mu_1(y)dy\right. \\ &\quad + \int_{t-x-h}^\infty \phi_1(y-t+x+h)e^{-\int_{y-t+x+h}^y \mu_1(\tau)d\tau}\mu_1(y)dy\end{aligned}$$

$$-\int_0^{t-x} p_1(0,t-x-y)e^{-\int_0^y \mu_1(\tau)d\tau}\mu_1(y)dy - \int_{t-x}^{\infty} \phi_1(y-t+x)e^{-\int_{y-t+x}^{y} \mu_1(\tau)d\tau}\mu_1(y)dy\bigg|$$

$$= \overline{p}\left|\int_0^{t-x-h} \lambda p_0(t-x-h-y)e^{-\int_0^y \mu_1(\tau)d\tau}\mu_1(y)dy\right.$$

$$+\int_0^{\infty} \phi_1(u)e^{-\int_u^{u+t-x-h} \mu_1(\tau)d\tau}\mu_1(u+t-x-h)du$$

$$\left.-\int_0^{t-x} \lambda p_0(t-x-y)e^{-\int_0^y \mu_1(\tau)d\tau}\mu_1(y)dy - \int_0^{\infty} \phi_1(u)\mu_1(u+t-x)e^{-\int_u^{u+t-x} \mu_1(\tau)d\tau}du\right|$$

$$\le \overline{p}\left|\int_0^{t-x-h} \lambda\phi_0 e^{-\lambda(t-x-h-y)}e^{-\int_0^y \mu_1(\tau)d\tau}\mu_1(y)dy\right.$$

$$\left.-\int_0^{t-x} \lambda\phi_0 e^{-\lambda(t-x-y)}\mu_1(y)e^{-\int_0^y \mu_1(\tau)d\tau}dy\right|$$

$$+\overline{p}\left|\int_0^{\infty} \phi_1(u)\left[\mu_1(u+t-x-h)e^{-\int_u^{u+t-x-h} \mu_1(\tau)d\tau} - \mu_1(u+t-x)e^{-\int_u^{u+t-x} \mu_1(\tau)d\tau}\right]dx\right|$$

$$= \lambda\overline{p}|\phi_0|\left|\int_0^{t-x-h} \mu_1(y)e^{-\lambda(t-x-h-y)-\int_0^y \mu_1(\tau)d\tau}dy\right.$$

$$-\int_0^{t-x-h} \mu_1(y)e^{-\lambda(t-x-y)-\int_0^y \mu_1(\tau)d\tau}dy$$

$$\left.-\int_0^{t-x} \mu_1(y)e^{-\lambda(t-x-y)-\int_0^y \mu_1(\tau)d\tau}dy + \int_0^{t-x-h} \mu_1(y)e^{-\lambda(t-x-y)-\int_0^y \mu_1(\tau)d\tau}\right|$$

$$+\overline{p}\left|\int_0^{\infty} \phi_1(u)\left[\mu_1(u+t-x-h)e^{-\int_u^{u+t-x-h} \mu_1(\tau)d\tau} - \mu_1(u+t-x-h)e^{-\int_u^{u+t-x} \mu_1(\tau)d\tau}\right.\right.$$

$$\left.\left.+\mu_1(u+t-x-h)e^{-\int_u^{u+t-x} \mu_1(\tau)d\tau} - \mu_1(u+t-x)e^{-\int_u^{u+t-x} \mu_1(\tau)d\tau}\right]du\right|$$

$$\le \lambda\overline{p}|\phi_0|\left\{\int_0^{t-x-h} \mu_1(y)\left|e^{-\lambda(t-x-h-y)-\int_0^y \mu_1(\tau)d\tau} - e^{-\lambda(t-x-y)-\int_0^y \mu_1(\tau)d\tau}\right|dy\right.$$

$$\left.+\int_{t-x-h}^{t-x} \mu_1(y)e^{-\lambda(t-x-y)-\int_0^y \mu(\tau)d\tau}dy\right\}$$

$$+\overline{p}\int_0^{\infty} |\phi_1(u)|\mu_1(u+t-x-h)\left|e^{-\int_u^{u+t-x-h} \mu_1(\tau)d\tau} - e^{-\int_u^{u+t-x} \mu_1(\tau)d\tau}\right|du$$

$$+\overline{p}\int_0^{\infty} |\phi_1(u)||\mu_1(u+t-x-h) - \mu_1(u+t-x)|e^{-\int_u^{u+t-x} \mu_1(\tau)d\tau}du$$

$$\le \lambda\overline{p\mu}\|\phi\|_X\left\{\int_0^{t-x-h} e^{-\lambda(t-x-y)-\int_0^y \mu_1(\tau)d\tau}\left|e^{-\lambda h}-1\right|du + \overline{\mu}|h|\right\}$$

$$+\overline{p\mu}\sup_{u\in[0,\infty)}\left|e^{-\int_u^{u+t-x-h} \mu_1(\tau)d\tau} - e^{-\int_u^{u+t-x} \mu_1(\tau)d\tau}\right|\|\phi\|_X + \overline{p}|h|\|\phi\|_X$$

$$\to 0, \quad \text{as} \quad h \to 0, \quad \text{uniformly for} \quad \phi. \quad (41)$$

$$\Longrightarrow \sum_{i=1}^{2} \int_0^t |p_i(0, t-x-h) - p_i(0, t-x)| e^{-\int_0^x \mu_i(\tau)\mathrm{d}\tau} \mathrm{d}x \to 0,$$

$$\text{as} \quad h \to 0, \ \text{uniformly for} \quad \phi. \quad (42)$$

Therefore, for $x \in (0,t),\ x+h \in (0,t)$, by (39), (42) and (36) we know

$$\sum_{i=1}^{2} \int_0^t |p_i(x+h,t) - p_i(x,t)| \mathrm{d}x \to 0, \quad \text{as} \quad h \to 0, \ \text{uniformly for} \quad \phi. \quad (43)$$

If $h \in (-t, 0), x \in [0,t]$, then from $p_i(x+h, 0) = 0$ for $x+h < 0, i = 1, 2$, we deduce

$$\sum_{i=1}^{2} \int_0^t |p_i(x+h,t) - p_i(x,t)| \mathrm{d}x$$

$$= \sum_{i=1}^{2} \left\{ \int_{-h}^t |p_i(x+h,t) - p_i(x,t)| \mathrm{d}x + \int_0^{-h} |p_i(x+h,t) - p_i(x,t)| \mathrm{d}x \right\}$$

$$= \sum_{i=1}^{2} \int_{-h}^t |p_i(x+h,t) - p_i(x,t)| \mathrm{d}x + \sum_{i=1}^{2} \int_0^{-h} |p_i(x+h,t) - p_i(x,t)| \mathrm{d}x. \quad (44)$$

Since $x+h \in [0,t)$, for $x \in [-h,t),\ h \in [-t,0)$, similar way to (37)-(42), it follows that

$$\sum_{i=1}^{2} \int_{-h}^t |p_i(x+h,t) - p_i(x,t)| \mathrm{d}x \to 0, \quad \text{as} \quad h \to 0, \text{uniformly for} \quad \phi, \quad (45)$$

$$\sum_{i=2}^{2} \int_0^{-h} |p_i(x+h,t) - p_i(x,t)| dx \to 0 \quad \text{as} \quad |h| \to 0, \ \text{uniformly for} \quad \phi. \quad (46)$$

By summing up (44)-(46), we obtain, for $x \in (0,t),\ h \in (-t,0)$,

$$\sum_{i=1}^{2} \int_0^t |p_i(x+h,t) - p_i(x,t)| \mathrm{d}x \to 0, \quad |h| \to 0. \quad (47)$$

(43) and (47) show that the result of this theorem holds. □

Theorem 5. *Assume that there exist two positive constants* $\overline{\mu}$, $\underline{\mu}$, *such that* $0 < \underline{\mu} \leq \mu(x) \leq \overline{\mu} < \infty$, *then*

$$\|W(t)\phi\|_X \leq e^{-\min\{\lambda, \underline{\mu}\}t} \|\phi\|_X.$$

Proof. For any $\phi \in X$, from the definition of $W(t)$ and Lemma 1, we estimate

$$\begin{aligned}
\|W(t)\phi\| &= |p_0| + \sum_{i=1}^{2}\int_t^\infty |p_i(x,t)|\mathrm{d}x \\
&\le |\phi_0|e^{-\lambda t} + \sum_{i=1}^{2}\int_t^\infty |\phi_i(x-t)|e^{-\int_{x-t}^{x}\mu_i(\tau)\mathrm{d}\tau}\mathrm{d}x \\
&\le |\phi_0|e^{-\lambda t} + \sum_{i=1}^{2}\int_0^\infty |\phi_i(y)|e^{-\int_{y}^{y+t}\underline{\mu}\mathrm{d}\tau}\mathrm{d}x \\
&\le |\phi_0|e^{-\lambda t} + \sum_{i=1}^{2}\|\phi_i\|_{L^1[0,\infty)}e^{-\underline{\mu}t} \le e^{-\min\{\lambda,\underline{\mu}\}t}\|\phi\|_X. \quad (48)
\end{aligned}$$

(48) shows that result of this theorem is right. □

From Theorem 5 we derive

$$\|S(t) - Z(t)\| = \|W(t)\| \le e^{-\min\{\lambda,\underline{\mu}\}t}.$$

From which together with Definition 2.7 in Nagel [5], we deduce the following result.

Theorem 6. *If $\mu_i(x)$ are Lipschitz continuous and satisfy $0 < \underline{\mu} \le \mu_i(x) \le \overline{\mu} < \infty$ $(i = 1, 2)$, then $S(t)$ is a quasi-compact operator in X.*

Since $U : X \to R$ is a bounded linear operator (see the third step in Theorem 1), U is a compact operator, from Theorem 6 and Proposition 2.9 in Nagel [5], p. 215, we conclude the following result.

Corollary 1. *If $\mu_i(x)$ are Lipschitz continuous and satisfy $0 < \underline{\mu} \le \mu_i(x) \le \overline{\mu} < \infty$ $(i = 1, 2)$, then $T(t)$ is a quasi-compact operator in X.*

In Guo et al [1], the author obtained that 0 is an eigenvalue with algebraic multiplicity one, so we deduce:

Theorem 7. *If $\mu_i(x)$ are Lipschitz continuous and satisfy $0 < \underline{\mu} \le \mu_i(x) \le \overline{\mu} < \infty$ $(i = 1, 2)$, then there exist a positive projection operator P and positive numbers $\delta > 0$, $m > 0$ such that*

$$\|T(t) - P\| \le me^{-\delta t}, \quad \forall t \in [0, \infty),$$

where $P = \frac{1}{2\pi i}\int_{\overline{\Gamma}}(zI - A - U)^{-1}dz$, $\overline{\Gamma}$ is a circle with center 0 and sufficient small radius.

From Corollary 1, Guo et al [1] and Nagel [5], p. 216 we know that $\{\gamma \in \sigma(A+U) \mid \mathrm{Re}\gamma = 0\} = \{0\}$. In other words, all points on the imaginary axis except for zero belong to the resolvent set of $A+U$. Thus by using Theorem 14 in Gupur et al [3] we obtain the following result, which was obtained in Guo et al [1].

Theorem 8. *If $\mu_i(x)$ are Lipschitz continuous and satisfy $0 < \underline{\mu} \le \mu_i(x) \le \overline{\mu} < \infty$ $(i = 1, 2)$, then the time-dependent solution of the system (7)-(8) converges strongly to the steady-state solution of the system (7)-(8), that is,*

$$\lim_{t\to\infty} \|p(\cdot, t) - p(\cdot)\| = 0,$$

where $p(x)$ is the eigenvector corresponding to 0.

In the following we will discuss further results when $\mu_i(x) = \mu_i$ $(i = 1, 2)$, here μ_i are constants.

Lemma 2. *$A+U$ has no other eigenvalues in the left half complex plane.*

Proof. Consider equations $(\gamma I - A - U)p = 0$. This is equivalent to

$$(\gamma + \lambda)p_0 = \mu_1 q \int_0^\infty p_1(x)\mathrm{d}x + \mu_2 \int_0^\infty p_2(x)\mathrm{d}x, \tag{49}$$

$$\frac{\mathrm{d}p_1(x)}{\mathrm{d}x} = -(\gamma + \mu_1)p_1(x), \tag{50}$$

$$\frac{\mathrm{d}p_2(x)}{\mathrm{d}x} = -(\gamma + \mu_2)p_2(x), \tag{51}$$

$$p_1(0) = \lambda p_0, \tag{52}$$

$$p_2(0) = \mu_1 \overline{p} \int_0^\infty p_1(x)\mathrm{d}x. \tag{53}$$

Solving (50) and (51) we have

$$p_1(x) = a_1 e^{-(\gamma+\mu_1)x}, \tag{54}$$

$$p_2(x) = a_2 e^{-(\gamma+\mu_2)x}. \tag{55}$$

By combining (54) and (55) with (52) and (53) we deduce for $\mathrm{Re}\gamma > \max\{-\mu_1, -\mu_2\}$

$$a_1 = p_1(0) = \lambda p_0, \tag{56}$$

$$a_2 = p_2(0) = \mu_1 \overline{p} \int_0^\infty a_1 e^{-(\gamma+\mu_1)x}\mathrm{d}x = -\frac{\mu_1 \overline{p}}{\gamma + \mu_1} a_1. \tag{57}$$

By inserting (54)-(57) into (49) and noting $p_0 \neq 0$ we derive

$$(\gamma + \lambda)p_0 = \frac{\mu_1 q \lambda}{\gamma + \mu_1} p_0 + \frac{\mu_1 \mu_2 \overline{p} \lambda}{(\gamma + \mu_1)(\gamma + \mu_2)} p_0$$

$$\Longrightarrow (\gamma + \lambda)(\gamma + \mu_1)(\gamma + \mu_2) = \mu_1 q \lambda (\gamma + \mu_2) + \mu_1 \mu_2 \overline{p} \lambda$$

$$\Longrightarrow \gamma \left[\gamma^2 + (\mu_1 + \mu_2 + \lambda)\gamma + (\mu_1\mu_2 + \mu_1\lambda + \mu_2\lambda - \mu_1 q \lambda)\right] = 0$$

$$\Rightarrow \gamma_1 = 0,$$
$$\gamma_2 = \frac{-(\mu_1+\mu_2+\lambda)+\sqrt{\mu_1^2+\mu_2^2+\lambda^2-2(\mu_1\lambda+\mu_2\lambda+\mu_1\mu_2)+4\mu_1 q\lambda}}{2},$$
$$\gamma_3 = \frac{-(\mu_1+\mu_2+\lambda)-\sqrt{\mu_1^2+\mu_2^2+\lambda^2-2(\mu_1\lambda+\mu_2\lambda+\mu_1\mu_2)+4\mu_1 q\lambda}}{2}.$$

$\gamma_1 = 0$ is just the result in Guo et al [1]. Hence, in the following we will discuss only γ_2 and γ_3. By using the relation $\overline{p}+q=1$ we have

$$\begin{aligned}\gamma_2 &= \frac{-(\mu_1+\mu_2+\lambda)+\sqrt{\mu_1^2+\mu_2^2+\lambda^2-2(\mu_1\lambda+\mu_2\lambda+\mu_1\mu_2)+4\mu_1 q\lambda}}{2}\\ &= \frac{-(\mu_1+\mu_2+\lambda)+\sqrt{(\mu_2-\mu_1-\lambda)^2-4\mu_1\lambda+4\mu_1 q\lambda}}{2}\\ &= \frac{-(\mu_1+\mu_2+\lambda)+\sqrt{(\mu_2-\mu_1-\lambda)^2-4\mu_1\overline{p}\lambda}}{2}\\ &< \frac{-(\mu_1+\mu_2+\lambda)+\sqrt{(\mu_2-\mu_1-\lambda)^2}}{2}.\end{aligned}$$

If $\mu_2-\mu_1-\lambda \geq 0$, then

$$\gamma_2 < \frac{-(\mu_1+\mu_2+\lambda)+(\mu_2-\mu_1-\lambda)}{2} = -\mu_1-\lambda < -\mu_1. \tag{58}$$

If $\mu_2-\mu_1-\lambda < 0$, then

$$\gamma_2 < \frac{-(\mu_1+\mu_2+\lambda)-(\mu_2-\mu_1-\lambda)}{2} = -\mu_2. \tag{59}$$

From (58) and (59) together with $\mathrm{Re}\gamma > \max\{-\mu_1,-\mu_2\}$ we know that γ_2 is not an eigenvalue of $A+U$. In similar way, it is not difficult to prove that γ_3 is not an eigenvalue of $A+U$. So the result of this lemma is right. □

Lemma 3. *If $z \in \rho(A+U)$, then we have*

$$(zI-A-U)^{-1}y(x) = (p_0, p_1(x), p_2(x)), \quad \forall y \in X,$$

where

$$\begin{aligned}p_0 = &\left\{\frac{1}{z+\lambda}+\frac{\lambda\mu_1 qz+\lambda\mu_1\mu_2}{z(z+\lambda)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right\}y_0+\\ &\left\{\frac{\mu_1 q}{(z+\lambda)(z+\mu_1)}+\frac{\mu_1 q(\lambda\mu_1 qz+\lambda\mu_1\mu_2)}{z(z+\lambda)(z+\mu_1)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right.\\ &\left.+\frac{\mu_1\mu_2\overline{p}(\lambda\mu_1 qz+\lambda\mu_1\mu_2)+\mu_1\mu_2\overline{p}z[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}{z(z+\lambda)(z+\mu_1)(z+\mu_2)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right\}\\ &\times\int_0^\infty y_1(x)dx\end{aligned}$$

$$+\left\{\frac{\mu_2}{(z+\lambda)(z+\mu_2)}+\frac{\lambda\mu_1\mu_2 q}{z(z+\lambda)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right.$$
$$\left.+\frac{\lambda\mu_1\mu_2^2\overline{p}}{z(z+\lambda)(z+\mu_2)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right\}\int_0^\infty y_2(x)dx,$$

$$p_1(x)=p_1(0)e^{-(z+\mu_1)x}+e^{-(z+\mu_1)x}\int_0^x e^{(z+\mu_1)\tau}y_1(\tau)\mathrm{d}\tau,$$
$$p_2(x)=p_2(0)e^{-(z+\mu_2)x}+e^{-(z+\mu_2)x}\int_0^x e^{(z+\mu_2)\tau}y_2(\tau)\mathrm{d}\tau,$$

$$p_1(0)=\frac{\lambda(z+\mu_1)(z+\mu_2)}{z[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}y_0$$
$$+\frac{\lambda\mu_1 qz+\lambda\mu_1\mu_2}{z[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\int_0^\infty y_1(x)dx$$
$$+\frac{\lambda\mu_2(z+\mu_1)}{z[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\int_0^\infty y_2(x)dx,$$

$$p_2(0)=\frac{\lambda\mu_1\overline{p}(z+\mu_2)}{z[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}y_0$$
$$+\frac{\mu_1\overline{p}(\lambda\mu_1 qz+\lambda\mu_1\mu_2)+\mu_1\overline{p}z[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}{z(z+\mu_1)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}$$
$$\times\int_0^\infty y_1(x)dx+\frac{\lambda\mu_1\mu_2\overline{p}}{z[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\int_0^\infty y_2(x)dx.$$

Proof. For $\forall y\in X$, consider $(zI-A-U)p=y$, that is,

$$(z+\lambda)p_0=\mu_1 q\int_0^\infty p_1(x)\mathrm{d}x+\mu_2\int_0^\infty p_2(x)\mathrm{d}x+y_0, \tag{60}$$
$$\frac{\mathrm{d}p_1(x)}{\mathrm{d}x}=-(z+\mu_1)p_1(x)+y_1(x), \tag{61}$$
$$\frac{\mathrm{d}p_2(x)}{\mathrm{d}x}=-(z+\mu_2)p_2(x)+y_2(x), \tag{62}$$
$$p_1(0)=\lambda p_0, \tag{63}$$
$$p_2(0)=\mu_1\overline{p}\int_0^\infty p_1(x)\mathrm{d}x. \tag{64}$$

By solving (60)-(62) we have

$$p_0=\frac{\mu_1 q\int_0^\infty p_1(x)\mathrm{d}x+\mu_2\int_0^\infty p_2(x)\mathrm{d}x+y_0}{z+\lambda}, \tag{65}$$
$$p_1(x)=a_3e^{-(z+\mu_1)x}+e^{-(z+\mu_1)x}\int_0^x e^{(z+\mu_1)\tau}y_1(\tau)\mathrm{d}\tau, \tag{66}$$

$$p_2(x) = a_4 e^{-(z+\mu_2)x} + e^{-(z+\mu_2)x}\int_0^x e^{(z+\mu_2)\tau} y_2(\tau)\mathrm{d}\tau. \tag{67}$$

Inserting (66) and (67) into (65) and using $a_i = p_i(0),\ i = 3, 4$, gives

$$p_0 = \frac{y_0}{z+\lambda} + \frac{\mu_1 q\left[p_1(0) + \int_0^\infty y_1(x)\mathrm{d}x\right]}{(z+\lambda)(z+\mu_1)} + \frac{\mu_2\left[p_2(0) + \int_0^\infty y_2(x)\mathrm{d}x\right]}{(z+\lambda)(z+\mu_2)}, \tag{68}$$

$$p_1(x) = p_1(0)e^{-(z+\mu_1)x} + e^{-(z+\mu_1)x}\int_0^x e^{(z+\mu_1)\tau} y_1(\tau)\mathrm{d}\tau, \tag{69}$$

$$p_2(x) = p_2(0)e^{-(z+\mu_2)x} + e^{-(z+\mu_2)x}\int_0^x e^{(z+\mu_2)\tau} y_2(\tau)\mathrm{d}\tau. \tag{70}$$

By combining (68)-(70) with (63)-(64) and using the Fubini Theorem it follows that

$$\begin{aligned}
p_1(0) = \lambda p_0 = {}& \frac{\lambda}{z+\lambda}y_0 + \frac{\mu_1 q\lambda}{(z+\lambda)(z+\mu_1)}p_1(0) + \frac{\mu_1 q\lambda}{(z+\lambda)(z+\mu_1)}\int_0^\infty y_1(x)\mathrm{d}x \\
& + \frac{\mu_2\lambda}{(z+\lambda)(z+\mu_2)}p_2(0) + \frac{\mu_2\lambda}{(z+\lambda)(z+\mu_2)}\int_0^\infty y_2(x)\mathrm{d}x \\
& \Longrightarrow \frac{z^2 + (\mu_1+\lambda)z + \mu_1\lambda\overline{p}}{(z+\lambda)(z+\mu_1)}p_1(0) - \frac{\mu_2\lambda}{(z+\lambda)(z+\mu_2)}p_2(0) \\
= {}& \frac{\lambda}{z+\lambda}y_0 + \frac{\mu_1 q\lambda}{(z+\lambda)(z+\mu_1)}\int_0^\infty y_1(x)\mathrm{d}x + \frac{\mu_2\lambda}{(z+\lambda)(z+\mu_2)}\int_0^\infty y_2(x)\mathrm{d}x,
\end{aligned} \tag{71}$$

$$\begin{aligned}
p_2(0) = \mu_1\overline{p}\int_0^\infty p_1(x)\mathrm{d}x = {}& \mu_1\overline{p}\frac{p_1(0) + \int_0^\infty y_1(x)\mathrm{d}x}{z+\mu_1} \\
= {}& \frac{\mu_1\overline{p}}{z+\mu_1}p_1(0) + \frac{\mu_1\overline{p}}{z+\mu_1}\int_0^\infty y_1(x)dx.
\end{aligned} \tag{72}$$

From (71) and (72) it is not difficult to determine

$$\begin{aligned}
p_1(0) = {}& \frac{\lambda(z+\mu_1)(z+\mu_2)}{z[z^2 + (\lambda+\mu_1+\mu_2)z + \lambda\mu_1\overline{p} + \lambda\mu_2 + \mu_1\mu_2]}y_0 \\
& + \frac{\lambda\mu_1 qz + \lambda\mu_1\mu_2}{z[z^2 + (\lambda+\mu_1+\mu_2)z + \lambda\mu_1\overline{p} + \lambda\mu_2 + \mu_1\mu_2]}\int_0^\infty y_1(x)dx \\
& + \frac{\lambda\mu_2(z+\mu_1)}{z[z^2 + (\lambda+\mu_1+\mu_2)z + \lambda\mu_1\overline{p} + \lambda\mu_2 + \mu_1\mu_2]}\int_0^\infty y_2(x)dx.
\end{aligned} \tag{73}$$

$$\begin{aligned}
p_2(0) = {}& \frac{\lambda\mu_1\overline{p}(z+\mu_2)}{z[z^2 + (\lambda+\mu_1+\mu_2)z + \lambda\mu_1\overline{p} + \lambda\mu_2 + \mu_1\mu_2]}y_0 \\
& + \frac{\mu_1\overline{p}(\lambda\mu_1 qz + \lambda\mu_1\mu_2) + \mu_1\overline{p}z[z^2 + (\lambda+\mu_1+\mu_2)z + \lambda\mu_1\overline{p} + \lambda\mu_2 + \mu_1\mu_2]}{z(z+\mu_1)[z^2 + (\lambda+\mu_1+\mu_2)z + \lambda\mu_1\overline{p} + \lambda\mu_2 + \mu_1\mu_2]}
\end{aligned}$$

$$\times\int_0^\infty y_1(x)dx+\frac{\lambda\mu_1\mu_2\overline{p}}{z[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\int_0^\infty y_2(x)dx. \tag{74}$$

$$p_0=\left\{\frac{1}{z+\lambda}+\frac{\lambda\mu_1 qz+\lambda\mu_1\mu_2}{z(z+\lambda)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right\}y_0+$$
$$\left\{\frac{\mu_1 q}{(z+\lambda)(z+\mu_1)}+\frac{\mu_1 q(\lambda\mu_1 qz+\lambda\mu_1\mu_2)}{z(z+\lambda)(z+\mu_1)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right.$$
$$\left.+\frac{\mu_1\mu_2\overline{p}(\lambda\mu_1 qz+\lambda\mu_1\mu_2)+\mu_1\mu_2\overline{p}z[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}{z(z+\lambda)(z+\mu_1)(z+\mu_2)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right\}$$
$$\times\int_0^\infty y_1(x)dx$$
$$+\left\{\frac{\mu_2}{(z+\lambda)(z+\mu_2)}+\frac{\lambda\mu_1\mu_2 q}{z(z+\lambda)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right.$$
$$\left.+\frac{\lambda\mu_1\mu_2^2\overline{p}}{z(z+\lambda)(z+\mu_2)[z^2+(\lambda+\mu_1+\mu_2)z+\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2]}\right\}\int_0^\infty y_2(x)dx. \tag{75}$$

Through inserting (73) and (74) into (69) and (70) respectively, and combining with (75) we obtain the desired result of this lemma. □

Lemma 4. *For $\forall y\in X$, P in Theorem 7 is given by*

$$Py(x)=\begin{pmatrix}\frac{\mu_1\mu_2}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}\left[y_0+q\int_0^\infty y_1(x)dx+\overline{p}\int_0^\infty y_2(x)dx\right]\\ \frac{\lambda\mu_1\mu_2}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}e^{-\mu_1 x}\left[y_0+\int_0^\infty y_1(x)dx+\int_0^\infty y_2(x)dx\right]\\ \frac{\lambda\mu_1\mu_2\overline{p}}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}e^{-\mu_2 x}\left[y_0+\int_0^\infty y_1(x)dx+\int_0^\infty y_2(x)dx\right]\end{pmatrix}.$$

Especially, for the initial value $p(0)=(1,0,0)$, we have

$$Pp(0)=\begin{pmatrix}\frac{\mu_1\mu_2}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}\\ \frac{\lambda\mu_1\mu_2}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}e^{-\mu_1 x}\\ \frac{\lambda\mu_1\mu_2\overline{p}}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}e^{-\mu_2 x}\end{pmatrix}=p(x),\ \text{the eigenvector corresponds to}\quad 0.$$

Proof. From Corollary 1 and Lemma 2 we obtain that 0 is a pole of $A+U$ of order one. Thus by using the residue theorem in complex analysis and Lemma 3 we have, for $\forall y\in X$,

$$Py(x)=\frac{1}{2\pi i}\int_\Gamma(zI-A-U)^{-1}y(x)dz=\lim_{z\to 0}z(zI-A-U)^{-1}y(x)$$

$$= \begin{pmatrix} \frac{\mu_1\mu_2}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}\left[y_0+\int_0^\infty y_1(x)dx+\int_0^\infty y_2(x)dx\right] \\ \frac{\lambda\mu_1\mu_2}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}e^{-\mu_1 x}\left[y_0+\int_0^\infty y_1(x)dx+\int_0^\infty y_2(x)dx\right] \\ \frac{\lambda\mu_1\mu_2\overline{p}}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}e^{-\mu_2 x}\left[y_0+\int_0^\infty y_1(x)dx+\int_0^\infty y_2(x)dx\right] \end{pmatrix}.$$

Especially, for the initial value $p(0)=(1,0,0)$ of the system (7)-(8), we deduce

$$Pp(0) = \begin{pmatrix} \frac{\mu_1\mu_2}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2} \\ \frac{\lambda\mu_1\mu_2}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}e^{-\mu_1 x} \\ \frac{\lambda\mu_1\mu_2\overline{p}}{\lambda\mu_1\overline{p}+\lambda\mu_2+\mu_1\mu_2}e^{-\mu_2 x} \end{pmatrix} = p(x), \quad \text{the eigenvector corresponding to} \quad 0.$$

The proof of this lemma is complete. □

Theorem 9. *The time-dependent solution $p(x,t)$ of the system (7)-(8) converges exponentially to the steady-state solution $p(x)$ of the system (7)-(8), that is, there exist $\delta > 0$ and $M > 0$ such that*

$$\|p(\cdot,t)-p(\cdot)\| \le Me^{-\delta t}, \quad \forall t \in [0,\infty).$$

Proof. By Theorem 3 we know that the time-dependent solution $p(x,t)$ of the system (7)-(8) can be expressed as

$$p(x,t) = T(t)p(0), \quad \forall t \in [0,\infty).$$

From this together with Theorem 7 and Lemma 4 we derive

$$\begin{aligned}\|p(\cdot,t)-p(\cdot)\| = \|T(t)p(0)-Pp(0)\| \le \|T(t)-P\|\|p(0)\| \le Me^{-\delta t}\|p(0)\| \\ = Me^{-\delta t}, \quad \forall t \in [0,\infty).\end{aligned}$$

Thus we complete the proof of Theorem 9. □

Theorem 3 and Theorem 7 imply structure of the time-dependent solution $p(x,t)$ of the system (7)-(8). Theorem 9 implies stability of the queueing system.

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