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BRILL-NOETHER THEORY FOR VECTOR BUNDLES ON STABLE CURVES WITH TWO COMPONENTS, A UNIQUE SINGULAR POINT AND BALANCED GENERA

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be a stable curve with two irreducible components, $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2) = 1$ and $p_a(X_1) = p_a(X_2)$. Here we construct several ω_X-stable vector bundles on X with many sections in terms of the gonality of X_1 and X_2.

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1. Introduction

Here we consider the Brill-Noether theory for vector bundles on the following reducible curve. Let X be a projective and nodal curve with two irreducible components X_1, X_2 such that $\sharp(X_1 \cap X_2) = 1$. It is easy to use the Brill-Noether theory for line bundles on the components X_1 and X_2 to get several vector bundles on X with many section (see Propositions 1 and 2). For smooth curves this approach was used in [1] and gave many stable vector bundles. To speak about stability or semistability for sheaves on X we need to fix a polarization ([5], p. 153). As in [2], [3] and [4] we use the canonical polarization. The vector bundles we constructed are ω_X-semistable only if $|p_a(X_1) - p_a(X_2)| \leq 1$ and for most statements we need to assume $p_a(X_1) = p_a(X_2)$. As an immediate consequence of our results we get the following results.

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Corollary 1. *Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2) = 1$. Assume $p_a(X_1) = p_a(X_2) > 0$. Fix integers $r \geq 3$, $k > 0$, and assume the existence of $M_i \in \mathrm{Pic}^{k-1}(X_i)$, $i = 1, 2$, such that $a_i := h^0(X_i, M_i) \geq 2$. Then there exists a rank r ω_X-stable vector bundle E on X such that $h^0(X, E) \geq r(a_1 + a_2 - 1)$ and E has multidegree $(rk + 1, rk)$.*

Corollary 2. *Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2) = 1$ and both X_1 and X_2 are smooth. Assume $p_a(X_1) = p_a(X_2) > 0$. Fix an integer $k > 0$, 2 non-isomorphic line bundles $R_{1,j}$, $j = 1, 2$, and 2 non-isomorphic line bundles $R_{2,j}$ on X_2, $j = 1, 2$, such that $\deg(R_{i,j}) = k$ for all i, j. Set $a_{i,j} := h^0(X_i, R_{i,j})$. Fix an integer $c \geq k + 1$. Then there exists a rank two ω_X-stable vector bundle E on X such that $h^0(X, E) \geq -2 + \sum_{i=1}^{2} \sum_{j=1}^{2} a_{i,j}$ and E has multidegree (c, c).*

Remark 1. Fix integers $a \geq 2$ and $k \geq 3$. Let C be a smooth and connected curve of genus $q > 0$ such that there is $R \in \mathrm{Pic}^{k-1}(C)$ such that $h^0(C, R) \geq a$. Since $q > 0$, the line bundles $R(Q)$, $Q \in C$, are pairwise non-isomorphic. Hence they be used (if $X_1 = C$) in the statements of Corollary 2.

For the ω_X-stability of rank 2 sheaves on X with depth 1, but not locally free, see Proposition 4.

2. More Statements and the Proofs

Definition 1. Fix a reduced projective curve Y, an irreducible component T of Y, $Q \in T \cap Y_{reg}$ and a rank r vector bundle E on Y. Let $\mathbb{K}_Q$ denote the skyscraper sheaf supported by Q and with $h^0 = 1$. A vector bundle F on Y is said to be obtained from F making a positive elementary transformation supported by Q if it fits in an exact sequence of coherent sheaves on Y:

$$0 \to E \to F \to \mathbb{K}_Q \to 0\,. \tag{1}$$

Notice that $\mathrm{rank}(F) = r$, $\deg(F) = \deg(E) + 1$, $\deg(F|T) = \deg(E|T) + 1$ and $F|D \cong E|D$ for every irreducible component $D \neq T$ of Y. The set of all isomorphism classes of vector bundles obtained from E making a positive elementary supported by Q is non-empty and it is parametrized (perhaps not even finite-to-one) by a projective space of dimension $r - 1$.

Remark 2. Let X be a projective curve such that $X = A_1 \cup A_2$ with A_1, A_2 proper subcurves and $A_1 \cap A_2$ a unique point $P \in X$. Assume that P is an ordinary node of X. Fix $R_i \in \mathrm{Pic}(A_i)$, $i = 1, 2$. Then there exists a unique $L \in \mathrm{Pic}(X)$ such that $L|A_1 \cong R_1$ and $L|A_2 \cong R_2$.

Lemma 1. *Let X be a projective curve such that $X = A_1 \cup A_2$ with A_1, A_2 proper subcurves and $A_1 \cap A_2$ a unique point $P \in X$. Assume that P is an ordinary node of X. Fix a rank r vector bundle on X. Then $h^0(X, E) \geq h^0(A_1, E|A_1) + h^0(A_2, E|A_2) - r$ and $h^1(X, E) \geq h^1(A_1, E|A_1) + h^1(A_2, E|A_2)$. If $E|A_1$ is spanned at P, then $h^0(X, E) = h^0(A_1, E|A_1) + h^0(A_2, E|A_2) - r$ and $h^1(X, E) = h^1(A_1, E|A_1) + h^1(A_2, E|A_2)$.*

Proof. Use the long cohomology exact sequence of the Mayer-Vietoris exact sequence

$$0 \to E \to E|A_1 \oplus E|A_2 \to E|\{P\} \to 0 \tag{2}$$

and the surjectivity of the restriction map $H^0(A_1, E|A_1) \to H^0(\{P\}, E|\{P\}) \cong \mathbb{K}^{\oplus r}$ if $E|A_1$ is spanned at P. □

Remark 3. Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2) = 1$. Assume $p_a(X_1) = p_a(X_2) > 0$. Let L be a line bundle of bidegree (d_1, d_2) on X. It is easy to check that X satifies Caporaso's Basic Inequality ([2], or [3], Definition 1.1), i.e. it is ω_X-semistable ([4], Theorem 10.3.1) if and only if $|d_1 - d_2| \leq 1$. Taking the restriction maps $L \to L|X_i$, $i = 1, 2$, we see that L is ω_X-stable if and only if $d_1 = d_2$.

Remark 4. Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2) = 1$. Assume $p_a(X_1) = p_a(X_2) - 1 > 0$. Let L be a line bundle of bidegree (a, a) on X. It is easy to check that X satifies Caporaso's Basic Inequality ([2], or [3], Definition 1.1), i.e. it is ω_X-semistable ([4], Theorem 10.3.1).

Proposition 1. *Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2)$, say $\{P\} = X_1 \cap X_2$. Assume $p_a(X_1) = p_a(X_2) > 0$. Fix integers $r \geq 2$, $k > 0$, $a_i \geq 2$, $b_i \geq 2$, $i = 1, 2$, and an integer e such that $0 \leq e \leq r$. Assume the existence of $R_i \in Pic^k(X_i)$ and $L_i \in Pic^{k+1}(X_i)$, $i = 1, 2$, such that $h^0(X_i, R_i) \geq a_i$, $h^0(X_i, L_i) \geq b_i$. There there exists an ω_X-semistable rank r vector bundle E on X such that E has multidegree $(rk+e, rk+r-e)$ and $h^0(X, E) \geq eb_1 + (r-e)a_1 + (r-e)a_2 + eb_2 - r$.*

Proof. If R_i is not spanned at P and a general section of R_i vanishes at P of order t we take $R_i' := R_i(tQ - tP)$ for a general $Q \in X_i$. In this way we reduce to the case in which each R_i and each L_i is spanned at P. Let A (resp. B) be the only line bundle on X such that $A|X_1 \cong R_1$ and $A|X_2 \cong L_2$ (resp. $B|X_1 \cong L_1$ and $B|X_2 \cong L_1$) (Remark 2). Set $E := A^{\oplus(r-a)} \oplus B^{\oplus e}$. Since each R_i and each L_i is spanned at P, we have $h^0(X, A) = h^0(X_1, R_1)+h^0(X_2, L_2)-1 \geq a_1+b_2-1$ and $h^0(X, B) = h^0(X_1, L_1) + h^0(X_2, R_2) - 1 \geq a_2 + b_1 - 1$ (Lemma 1). Hence $h^0(X, E) \geq eb_1+(r-e)a_1+(r-e)a_2+eb_2-r$. Since A has bidegree $(k, k+1)$ and B has bidegree $(k+1, k)$, both are ω_X-semistable and with the same ω_X-slope. Hence E is ω_X-semistable. □

Proposition 2. *Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2)$, say $\{P\} = X_1 \cap X_2$. Assume $p_a(X_1) = p_a(X_2)-1 > 0$. Fix integers $r \geq 2$, $k > 0$ and $a_i \geq 2$, $i = 1, 2$. Assume the existence of $R_i \in Pic^k(X_i)$ $i = 1, 2$, such that $h^0(X_i, R_i) \geq a_i$. Then there existse a rank r ω_X-semistable vector bundle E on X such that E has bidegree (rk, rk) and $h^0(X, E) \geq r(a_1 + a_2 - 1)$.*

Proof. Let A be the only line bundle on X such that $A|X_i \cong R_i$ for all i. Set $E := A^{\oplus r}$. □

Lemma 2. *Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2)$, say $\{P\} = X_1 \cap X_2$. Assume $q := p_a(X_1) = p_a(X_2)$. Let H be the polarization on X associated to the rational numbers $a_1 = a_2 = 1/2$. Fix $L \in Pic(X)$ with bidegree (k, k). Let R be a rank 1 subsheaf of L with multirank $(1, 0)$ or $(0, 1)$ of L. Then $\mu_H(R) \leq 2k-2q$.*

Proof. Assume for instance that R has multirank $(1, 0)$. Hence R is contained in the kernel, M, of the restriction map $L \to L|X_2$. SinceM/R has finite support, $\mu_H(R) \leq \mu_H(M)$. Since $X_1 \cap X_2 = \{P\}$ and P is an ordinary node, the $\mathcal{O}_X$-sheaf is isomorphic to a degree $k - 1$ line bundle on X. Hence $\chi(M) = k - q$. Thus $\mu_H(M) = 2k - 2q$. □

Theorem 1. *Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2)$, say $\{P\} = X_1 \cap X_2$. Assume $q := p_a(X_1) = p_a(X_2) > 0$. Fix integers $r \geq 2$, $k > 0$, $M_i \in Pic^{k-1}(X_i)$ and general $P_{i,j} \in X_i$, $i = 1, 2$, $1 \leq j \leq r$, and $Q_1 \in X_1\backslash\{P\}$. Set $R_{i,j} := M_i(P_{i,j}) \in Pic^k(X_i)$. Let L_j be the only line bundle on X such that $L_j|X_1 \cong R_{1,j}$ and $L_j|X_2 \cong R_{2,j}$. Set $F := \oplus_{j=1}^r L_j$. Let E be the general vector bundle on X obtained from F making a general positive elementary transformation supported by Q_1.*

(i) If $r \geq 3$, then E is ω_X-stable.

(ii) If $r = 2$, then E is ω_X-semistable, but not ω_X-stable.

Proof. Since $q = p_a(X_i) > 0$, the line bundles $R_{i,j}$, $1 \le j \le r$, are pairwise non-isomorphic. Until step (f) we fix r pairwise non-isomorphic line bundles $R_{1,j}$, $1 \le j \le r$, r pairwise non-isomorphic line bundles $R_{2,j}$, $1 \le j \le r$, and make the proof in this more general set-up. Only in step (f) we assume that the line bundles $R_{i,j}$ are the ones described in Theorem 1 for general $P_{i,j} \in X_i$, $i = 1, 2$, $1 \le i \le j$. Since $p_A(X_1) = p_a(X_2)$, the ω_X-polarization is obtained taking the polarization H determined by the rational numbers $a_1 := 1/2$ and $a_2 := 1/2$ in [5], p. 153. Each line bundle L_j exists and it is unique, up to isomorphisms (Remark 2). Each L_j is ω_X-stable (Remark 3). Since $\mu_H(L_j) = \mu_H(L_h)$ for all $j, h \in \{1, \dots, r\}$, F is H-polystable and its indecomposable factors are pairwise non-isomorphic stable line bundles with the same H-slope. Hence F is H-semistable.

(a) Let $\rho : E \to E|X_2$ be the restriction map. Set $U := \mathrm{Ker}(\rho)$. U is one of the depth 1 subsheaves of E which must be tested to see if E is H-stable or H-semistable. We have $\chi(U) = \deg(E) + r(1 - p_a(X)) - r(\chi(R_2)) = 2kr + 1 + r - 2rq - rk - r + rq = rk + 1 - rq$. Hence $\mu_H(G) = 2(rk + 1 - rq)/r = 2k + 2/r - 2q$. Hence if $r = 2$, $\mu_H(U) = \mu_H(E)$ and the vector bundle E is not H-stable, while $\mu_H(U) < \mu_H(E)$ if $r \ge 3$. Let $\rho_1 : E \to E|X_1$ be the restriction map. As above we see that $\mu_H(\mathrm{Ker}(\rho_1)) < \mu_H(E)$ for all $r \ge 2$.

(b) The vector bundle $E|X_1$ is stable ([1], Proposition 2.3).

(c) Here assume $r = 2$. We checked in part (a) that E is not H-stable. Assume that E is not H-semistable and take a proper subsheaf G of E with maximal H-slope. Hence $\mu_H(G) > \mu_H(E)$. Let r_i be the rank of G at a general point of X_i. Thus $0 \le r_1 \le 2$, $0 \le r_2 \le 2$, and $1 \le r_1 + r_2 \le 3$. By part (a) we have $r_1 > 0$ and $r_2 > 0$. First assume $r_1 = r_2 = 1$. Set $G' := G \cap F$. If $G = G'$, then $\mu_H(G) = \mu_H(G') \le \mu_H(F) < \mu_H(E)$, contradiction. Now assume $G \ne G'$. Hence $\mu_H(G) = \mu_H(G') + 1/2$. If $\mu_H(G') < \mu_H(F)$, then $\mu_H(G') \le \mu_H(F) - 1/2$ and hence $\mu_H(G) \le \mu_H(E)$, contradiction. Now assume $\mu_H(G') = \mu_H(F)$. Since F is H-polystable with L_1 and L_2 as its indecomposable factors, either $G' \cong L_1$ or $G' \cong L_2$. However, since L_1 and L_2 are not isomorphic, F has only two H-destabilizing subsheaves. For a general positive elementary transformations, each of them remains saturated in E. Hence $G' = G$, contradiction. Now assume $(r_1, r_2) = (1, 2)$. Hence E/G is a line bundle on X_1, which is a quotient of $E|X_1$. Hence $\chi(E/G) \ge k + 2 - q$ (step (b)). Hence $\chi(G) \le k - q$. Thus $\mu_H(G) < \mu_H(E)$, contradiction. Now assume $(r_1, r_2) = (2, 1)$. Hence E/G is a line bundle on X_2, which is a quotient of $E|X_2$. Hence $\mu_H(E/G) \ge \mu_H(U) = \mu_H(E)$ (step (a)), contradiction.

(d) By step (c) we may assume $r \ge 3$. F may be seen as a subsheaf of E

such that E/F is the length 1 skyscraper sheaf $\mathbb{K}_{Q_1}$ supported by Q_1. Hence $\mu_H(E) = \mu_H(F) + 1/r$. Assume that E is not H-stable and take a proper subsheaf G of E with maximal H-slope. The maximality of $\mu_H(G)$ implies that G is saturated in E, i.e. E/G has depth 1. Let r_i, $i = 1, 2$, be the rank of G at a general point of X_i. Taking as G a proper subsheaf with maximal H-slope and with minimal $r_1 + r_2$ we may assume that G is H-stable. Since G is a proper saturated subsgheaf of E, $1 \le r_1 + r_2 \le 2r - 1$. Set $G' := G \cap E$. Since F is H-semistable and $\mu_H(G) \ge \mu_H(E) > \mu_H(F)$, we have $G' \ne G$. Hence $G/G' \cong \mathbb{K}_{Q_1}$. Hence $\mu_H(G) = \mu_H(G') + 2/(r_1 + r_2)$. The semistability of F gives $\mu_H(G') \le \mu_H(F)$. Since $E|X_2 = F|X_1$, we have $r_1 > 0$. Set $F' := \oplus_{j=1}^{r-1} L_j$. Let E' be the general vector bundle on X obtained from F' making a positive elementary transformation supported by Q_1. Set $E'' := E' \oplus L_r$. By step (c) and induction on r we may assume that E' is H-semistable. Hence every subsheaf of E'' has H-slope at most $\mu_H(E'') = 1 + (r-1) + k + 1 - 2q$. Since E'' is obtained from E making a positive elementary transformation supported by Q_1, while E' is obtained from E making a general positive elementary transformation supported by Q_1, $\mu_H(G) \le 1/(r-1) + 2k + 1 - 2q$. First assume $\mu_H(G') = \mu_H(F)$. Since F is H-polystable and $(r_1, r_2) \ne (r, r)$, G' is a proper factor of F. Since the line bundles L_j, $1 \le j \le r$, are pairwise non-isomorphic, F has only finitely many direct factors (not just finitely many isomorphism classes of direct factors, as obvious by Krull-Schmidt theorem)). The generality of the positive elementary transformation giving E shows that every proper direct factor of F is saturated in E. Since $G' \ne G$, we got a contradiction. Hence $\mu_H(G') < \mu_H(F) = 2k + 1 - 2q$. Since $\mu_H(F)$ is an integer and G' has multirank (r_1, r_2), we get $\mu_H(G') \le 2k + 1 - 2q - 1/(r_1 + r_2)$. Hence $\mu_H(G) \le 2k + 1 - 2q + 1/(r_1 + r_2)$. Since $\mu_H(G) \ge \mu_H(E) = 2k + 1 - 2q + 1/r$, we get $r_1 + r_2 \le r$. Let $u_j : F \to L_j$, $1 \le j \le r$, denote the projection on the j-th factor. Let A be a non-zero subsheaf of G' with maximal H-slope. Hence $\mu_H(A) \ge \mu_H(G')$ with strict inequality if G' is not H-semistable. Let (c_1, c_2) be the multirank of A. Among all such subsheaves we take one for which the integer $c_1 + c_2$ is minimal. With this assumption A is H-stable. Since G' is a subsheaf of F, there is $j \in \{1, \dots, r\}$ such that $u_j(A) \ne 0$. Since A is stable, either $A \cong L_j$ and $u_j|A : A \to L_j$ is an isomorphism, or $\mu_H(A) < \mu_H(L_j) = 2k + 1 - 2q$ and $u_j(A)$ is a proper subsheaf of L_j.

(e) Here we assume that $u_j(A)$ is a proper subsheaf of L_j and call (b_1, b_2) its multirank. We have $(b_1, b_2) \in \{(1,1), (1,0), (0,1)\}$. First assume $(b_1, b_2) = (1, 1)$. Since $u_j(A)$ is a proper subsheaf of L_j, we have $\mu(u_j(A)) \le 2k - 2q$. Since A is H-stable, we get $\mu_H(A) \le 2k - 2q$. Hence $\mu_H(G) = 1/(r_1 + r_2) + \mu_H(G') \le 1/(r_1 + r_2) + \mu_H(A) \le 1/(r_1 + r_2) + 2k + 1 - 2q < \mu_H(E)$, contradiction. If

$(b_1, b_2) \in \{(0,1), (1,0)\}$, then the same proof works using Lemma 2.

(f) Now assume $u_j : |A : A \to L_j$ is an isomorphism, the uniqueness of the inclusion $L_j \hookrightarrow F$ gives that L_j is a direct factor of G', say $G' = G'' \otimes L_j$. Since L_j is saturated in E, we have $G'' \neq 0$. Let $B \subseteq \{1, \dots, r\}$ denote the set of all $j \in \{1, \dots, r\}$ such that L_j is a direct factor of G'. Set $b := \sharp(B)$. We saw that $b > 0$. Obviously, $b \leq \min\{r_1, r_2\}$. Hence $1 \leq b < r$. Now we take $R_{i,j} = M_i(P_{i,j})$. The set B may vary with the choice of the points. For general points the integer b is fixed. Hence it is fixed the subset B of $\{1, \dots, r\}$. Since $X_1^r \times X_2^r$ is irreducible, $B \neq \emptyset$ and $B \neq \{1, \dots, r\}$, we get a contradiction. □

Proposition 3. *Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2)$, say $\{P\} = X_1 \cap X_2$. Assume $q := p_a(X_1) = p_a(X_2) > 0$. Fix an integer $k > 0$, 2 pairwise non-isomorphic line bundles $R_{1,j} \in Pic^k(X_1)$, $j = 1, 2$,2 paiwise non-isomorphic line bundles $R_{2,j} \in Pic^k(X_2)$, $j = 1, 2$, and $Q_i \in X_i \backslash \{P\}$, $i = 1, 2$. Let L_j be the only line bundle on X such that $L_j|X_1 \cong R_{1,j}$ and $L_j|X_2 \cong R_{2,j}$. Set $F := L_1 \oplus L_2$. Let E be the general vector bundle on X obtained from F making a general positive elementary transformation supported by Q_1 and a general positive elementary transformation supported by Q_2. Then E is ω_X-stable.*

Proof. Since $p_A(X_1) = p_a(X_2)$, the ω_X-polarization is obtained taking the polarization H determined by the rational numbers $a_1 := 1/2$ and $a_2 := 1/2$ in [5], p. 153. Each line bundle L_j exists and it is unique, up to isomorphisms (Remark 2). Each L_j is ω_X-stable (Remark 3). Since $\mu_H(L_j) = \mu_H(L_h)$ for all $j, h \in \{1, \dots, r\}$, F is H-polystable and its indecomposable factors are pairwise non-isomorphic stable line bundles with the same H-slope. Hence F is H-semistable. Set $M_1 := L_1(Q_1)$, $M_2 := L_2(Q_2)$ and $E'' := M_1 \oplus M_2$. M_1 has multidegree $(k+1, k)$, while M_2 has multidegree $(k, k+1)$. Hence M_1 and M_2 are H-semistable (Remark 3). Note that $\mu_H(M_1) = \mu_H(M_2)$. Since M_1 and M_2 have different multidegrees, they are not isomorphic. Hence E'' is H-polystable and its indecomposable factors are pairwise non-isomorphic. Hence it has only two proper subsheaves with H-slope $\mu_H(E'') = 2k+2-2q$: $M_1 \oplus \{0\}$ and $\{0\} \oplus M_2$. Since E'' is obtained from F making two positive elementary transformations, one supported by Q_1 and one supported by Q_2, we get that E is H-semistable and, if not H-stable, at most finitely many proper subsheaves of it have slope $\mu_H(E)$, all of them having multidegree $(1, 1)$. Assume that E is not H-stable and take a proper subsheaf G of E with maximal H-slope. We just saw that G has type $(1, 1)$ and that $\mu_H(G) = \mu_H(E) = 2k + 1 - 2q$. The generality of the positive elementary transformations used to get E from F gives that E is indecomposable. Hence G is unique. Let E_i, $i = 1, 2$, be the vector bundle obtained from F making the same positive elementary transformation at Q_i we did to get E from F. Theorem 1 shows that each E_i is H-semistable. Set $G_i := G \cap E_i$, $i = 1, 2$. Hence $G' := G \cap F = G_1 \cap G_2$. Since $\mu_H(G_i) \le \mu_H(E_i)$, we get $\deg(G_i) = \deg(G) - 1 = 2k$. Hence $\deg(G \cap F) = 2k - 1$. Since L_1 and L_2 are not isomorphic, it is easy to check that they are saturated in E. Hence $G' \notin \{L_1, L_2\}$. Hence the inclusion $G' \hookrightarrow F$ gives two non-zero maps $v_i : G' \to L_i$, $i = 1, 2$. First assume that G is not locally free. Hence G' is not locally free. Since $v_i \neq 0$ and G' has multirank $(1, 1)$, $G' \cong \mathcal{I}_P \otimes L_i$. Since L_1 and L_2 are not isomorphic, this is absurd. Hence G and G' are locally free. The germ at P of any non-locally free depth 1 sheaf with pure rank 1 is isomorphic to the maximal ideal $m_{X,P}$ of the local ringt $\mathcal{O}_{X,P}$. Fix any two generators of $m_{X,P}$ as an $\mathcal{O}_{X,P}$-module and take the associated surjection $\rho : \mathcal{O}_{X,P}^{\oplus 2} \to m_{X,P}$. Since $\mathrm{Ker}(\rho)$ is not a free $\mathcal{O}_{X,P}$-module, $X \backslash \{P\}$ is smooth and the line bundle G is saturated in E, E/G is a line bundle. One of the line bundles G, E/G must have multidegree $(k+1, k)$, while the other one has multidegree $(k, k+1)$. Since E is indecomposable, exchanging the roles of X_1 and X_2 we get a contradiction. □

Proof of Corollary 1. Apply Theorem 1 and Lemma 1. □

Proof of Corollary 2. If $c = k + 1$, then apply Proposition 3. If $c \ge k + 2$, then apply Proposition 3 to the line bundles $R_{i,j}((c - k - 1)P_i)$ with P_i general in X_i and notice that if A is a subsheaf of a coherent sheaf B on X, then $h^0(X, B) \ge h^0(X, A)$. □

Proposition 4. *Let X be a nodal projective curve with two irreducible components, say $X = X_1 \cup X_2$, such that $\sharp(X_1 \cap X_2)$, say $\{P\} = X_1 \cap X_2$. Assume $p_a(X_1) = p_a(X_2) > 0$. Fix an integer $k > 0$, 2 non-isomorphic line bundles $R_{1,j} \in Pic^k(X_1)$, $j = 1, 2$, and 2 non-isomorphic line bundles $R_{2,j} \in Pic^k(X_2)$, $j = 1, 2$. Set $F := L_1 \oplus L_2$. Let E be a depth 1 sheaf on X fitting in an exact sequence*

$$0 \to F \to E \to \mathbb{K}_P \to 0 \tag{3}$$

Then E exists, it has pure rank 2 and it is ω_X-semistable.

Proof. The existence of E is obvious, because it exists locally and the local-to-global spectral sequence of the Ext-functors gives its global existence. Obviously, E has pure rank 2. Since $p_A(X_1) = p_a(X_2)$, the ω_X-polarization is obtained taking the polarization H determined by the rational numbers $a_1 := 1/2$

and $a_2 := 1/2$. Assume that E is not H-semistable and take a proper subsheaf G of E with maximal H-slope. Let (b_1, b_2) be the multirank of G. Taking $b_1 + b_2$ minimal among all proper subsheaves with H-slope $\mu_H(G)$ we may assume that G is H-stable. We have $(b_1, b_2) \in \{(1,0), (2,0), (1,1), (0,1), (0,2), (1,2)\}$. Set $G' := G \cap F$. Since F is H-semistable and $\mu_H(G) = (2k+3-2q)/2 = 1/2+\mu_H(F)$, $G' \neq G$. Hence $\mu_H(G) = \mu_H(G')+2/(b_1+b_2)$. If $\mu_H(G') < \mu_H(F)$, then $\mu_H(G') \leq \mu_H(F) - 1/(b_1 + b_2)$, because $\mu_H(F)$ is an integer. Hence the inequality $\mu_H(G) \geq \mu_H(F) + 1/2$ gives $b_1 + b_2 \leq 1$, i.e, $(b_1, b_2) \in \{(1,0), (0,1)\}$. Use that any rank 1 quotient of $E|X_i$ has degree at least $k+1$. □

Proposition 4 gives an obvious corollary related to the Brill-Noether theory of depth 1 sheaves on X with pure rank 2.

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