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REDUCIBLE SPACE CURVES IN POSITIVE CHARACTERISTIC

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we study in arbitrary characteristic reducible space curves with high arithmetic genus or high index of speciality.

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1. Introduction

Fix integers $d \geq n \geq 3$. A classical problem is the computation of the maximal arithmetic genus of an integral and non-degenerate curve $Y \subset \mathbb{P}^n$ with fixed degree birational onto its image and the study the Hilbert scheme of all curves whose arithmeticc genus is "very near" to the maximal one (Castelnuovo's theory (see [2], Chapter 3)). The same theory gives upper bounds for the integers $h^1(Y, \mathcal{O}_Y(t))$, $t \in \mathbb{N}$. Castelnuovo's theory is easier in characteristic zero, because in characteristic zero one can use the Uniform Position Principle ([2]; see [3] for the positive characteristic case). Here we consider the same problems for reducible curves in the case $n = 3$ (space curves). Everything we say here is either trivial or, at least, very easy in characteristic zero ([3] and [1]). Our starting point is the following observation.

Remark 1. Let $Y \subset \mathbb{P}^3$ be a reduced curve such that each of its irreducible component spans $\mathbb{P}^3$. Let $H \subset \mathbb{P}^3$ be a general plane. Assume that $H \cap Y$ is not in linearly general position, i.e. assume that $H \cap Y$ contains at least 3 collinear points. Since H is general, at least one of the following cases occurs:

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(i) there is $i \in \{1, \dots, s\}$ and an integer $z \geq 3$ such that a general secant line of C_i intersects C_i at exactly z points;

(ii) there are $i, j \in \{1, \dots, s\}$ such that $i \neq j$ and a general secant line of C_i intersects C_j;

(iii) $s \geq 3$ and there are integers i, j, h such that $1 \leq i < j < h \leq s$ and the line spanned by a general point of C_i and the general point of C_j intersects C_h.

Motivated from Remark 1 we introduce the following examples and definitions.

Example 1. Fix integers $n \geq 3$ and $s \geq 3$ and s integral curves $C_1, \dots, C_s \subset \mathbb{P}^n$. Assume $C_i \neq C_j$ for all $i \neq j$, that for every $i \in \{1, \dots, s\}$ the general secant line to C_i is not a multisecant to C_i and intersect no C_j with $j \neq i$, and that a general line intersecting $(C_1)_{reg} \backslash (C_1 \cap C_2)$ and $(C_2)_{reg} \backslash (C_1 \cap C_2)$ intersects every C_j, $j \in \{3, \dots, s\}$. Since for all $i \neq j$ the set $D(i,j)$ of all lines intersecting $(C_i)_{reg} \backslash (C_i \cap C_j)$ and $(C_j)_{reg} \backslash (C_i \cap C_j)$, and a general $D \in D(i,j)$ is not a secant line of C_i or of C_2, we get that a general $D(i,j)$ intersects each curve $C_1, \dots, C_s$ at a unique point. In particular $D(i,j) = D(1,2)$ for all $i \neq j$. Set $C := C_1 \cup \cdots \cup C_s$. Fix a general hyperplane $H \subset \mathbb{P}^n$. The generality of H gives that $H \cap C$ is formed by $\deg(C)$ points and for all $i \neq j$ and any $P_i \in C_i \cap H$ and any $P_j \in C_j \cap H$ the line $\langle \{P_i, P_j\} \rangle$ spanned by P_i and P_j intersects each $C_h \cap H$, $h \in \{1, \dots, s\}$, in exactly one point. Hence $\deg(C_h) = \deg(C_1)$ for all $h \in \{2, \dots, s\}$.

Example 2. Fix integers $y \geq 1$ and $x \geq 2$. Let $C_1, C_2 \subset \mathbb{P}^n$, $n \geq 3$, be integral curves such that $C_1 \neq C_2$, C_1 is not a line, and the general secant line of C_1 intersects C_1 at x points and C_2 at y points. Since there is an integral 2-dimensional family of such lines, we get that a general line intersecting both $C_1 \backslash (C_1 \cap C_2)$ and $C_2 \backslash (C_1 \cap C_2)$ contains exactly x points of C_1 and y points of C_2. Let $H \subset \mathbb{P}^n$ be a general hyperplane. Set $d_i := \deg(C_i)$, $i = 1, 2$. Let Σ denote the set of all lines of H spanned by two points of $H \cap C_1$. Let Σ' denote spanned by a point of $H \cap C_1$ and a point of $H \cap C_2$. The generality of H gives that H intersects transversally the curve $C := C_1 \cup C_2$, that each element of Σ contains exactly x points of $C_1 \cap H$ and y points of $C_2 \cap H$ and that each element of Σ' contains exactly x points of $C_1 \cap H$ and y points of $C_2 \cap H$. Since $x \geq 2$ we also get $\Sigma = \Sigma'$. We have $\sharp(\Sigma) = d_1(d_1 - 1)/x(x-1)$. We have $\sharp(\Sigma') = d_1 d_2 / xy$. Hence $d_2 = (d_1 - 1)/y$.

Example 1 is the case $y_1 = \cdots = y_s = 1$ of the next example.

Example 3. Fix integers $n \geq 3$, $s \geq 3$ and $y_i \geq 1$, $1 \leq i \leq s$. Fix s integral curves $C_1, \dots, C_s \subset \mathbb{P}^n$ such that $C_i \neq C_j$ for all $i \neq j$. For all

$i, j \in \{1, \dots, s\}$ let $D(i,j)$ be the set of all lines intersecting both $C_i \backslash (C_i \cap C_j)$ and $C_j \backslash (C_i \cap C_j)$. Each $D(i,j)$ is a non-empty and 2-dimensional integral quadi-projective variety. Assume the existence of $u, v \in \{1, \dots, s\}$ such that $u \neq v$ and the general e$D \in D(u,v)$ intersects eash Y_i at exactly y_i points. Since $y_i \geq 1$ for all i, we get that alll the $D(i,j)$ have a common open subset U and that each line $D \in U$ intersects each C_i at exactly y_i points. Fix a general hyperplane $H \subset \mathbb{P}^n$. The generality of H gives that $H \cap C$ is formed by $\deg(C)$ points and for all $i \neq j$ and any $P_i \in C_i \cap H$ and any $P_j \in C_j \cap H$ the line $\langle \{P_i, P_j\} \rangle$ spanned by P_i and P_j intersects each $C_h \cap H$, $h \in \{1, \dots, s\}$, in exactly y_i points. Hence $y_1 \cdot \deg(C_h) = y_h \cdot \deg(C_1)$ for all $h \in \{2, \dots, s\}$.

Definition 1. Fix integers d, x, y, r such that $r \geq 2$, $d > x \geq 2$, and $y \geq 1$. Fix a subset $S \subset \mathbb{P}^r$ such that $\sharp(S) = d$. We will say that S has the property $\clubsuit$ or that it satisfies $\clubsuit$ with respect to the data (r, d, x, y) or just the data (x, y) if there is a partition $S = S_1 \sqcup S_2$ such that S_1 spans $\mathbb{P}^r$ and the following condition is satisfied:

$\clubsuit$: Let Σ (resp. Σ') be the set of all lines of $\mathbb{P}^r$ spanned by 2 points of S_1 (resp. a point of S_1 and a point of S_2); then $\Sigma = \Sigma'$ and $\sharp(D \cap S_1) = x$, $\sharp(D \cap S_2) = y$ for every $D \in \Sigma$.

The bidegree of the partition $S = S_1 \sqcup S_2$ is the pair $(\sharp(S_1), \sharp(S_2))$.

Definition 2. Fix integers $r \geq 2$, $s \geq 3$ and $y_i \geq 1$, $1 \leq i \leq s$. Fix a subset $S \subset \mathbb{P}^r$ equipped with a partition $S = S_1 \sqcup \cdots \sqcup S_s$ such that each S_i spans $\mathbb{P}^r$ and the following condition is satisfied:

$\spadesuit$: For all $i, j \in \{1, \dots, s\}$ such that $i \neq j$ let $D(i,j)$ be the set of all lines spanned by a point of S_i and a point of S_j; then $D(i,j) = D(u,v)$ for every quadruple $(i, j, u, v) \in \{1, \dots, s\}^4$ such that $i \neq j$ and $u \neq v$ and $\sharp(D \cap S_h) = y_h$ for every $D \in D(1,2)$ and every $h \in \{1, \dots, s\}$.

The multidegree of the partition $S = S_1 \sqcup \cdots \sqcup S_s$ is the s-ple of integers $\sharp(S_1), \dots, \sharp(S_s))$. If $\spadesuit$ is satified, then we say that the partition $S = S_1 \sqcup \cdots \sqcup S_s$ has property $\spadesuit$ with respect to the s-ple $(y_1, \dots, y_s)$ or that it has $\spadesuit(y_1, \dots, y_s)$. $y_1 = \cdots = 1$, then we we say that the partition $S = S_1 \sqcup \cdots \sqcup S_s$ has property $\spadesuit*$.

Remark 2. Let $(d_1, \dots, d_s)$ be the multidegree of a partition $S = S_1 \sqcup \cdots \sqcup S_s$ satisfying $\spadesuit(y_1, \dots, y_s)$. We have $y_1 d_i = y_i d_1$ for every $i \in \{2, \dots, s\}$.

In the case $n = 3$ we are able just using elementary tools to get the following results.

Theorem 1. *Let $Y \subset \mathbb{P}^3$ be a reduced curve such that every irreducible component of Y spans $\mathbb{P}^3$. Set $d := \deg(Y)$. Then* $p_a(Y) \leq \lfloor d/2 \rfloor \lceil d/2 \rceil - d + 1$

and the equality holds iif and only if Y is contained in an irreducible quadric surface S and in a surface of degree $\lceil d/2 \rceil$ not containing S.

Proposition 1. *Let $Y \subset \mathbb{P}^3$ be a reduced curve such that each of its irreducible components are non-degenerate. Set $d := \deg(Y)$ and $m := \lfloor d/2 \rfloor + 1$. Then $h^1(Y, \mathcal{O}_Y(t)) \le \min\{0, \sum_{i=t}^{m-1}(d-2i-1)\}$ for all $t < m$.*

The story is very different if one allows that some of the irreducible components of Y are degenerate (see [4]).

2. The Proofs

Lemma 1. *Fix a finite subset $S \subset \mathbb{P}^2$ and an integer $k > 0$. Set $d := \sharp(S)$. Assume $d \le 2k+1$. There is a line $D \subset \mathbb{P}^2$ such that $\sharp(S \cap D) \ge k+2$ if and only if $h^1(\mathbb{P}^2, \mathcal{I}_S(k)) > 0$.*

Proof. Since the "if" part is obvious, we only need to prove the "only if" part. We use induction on k, the case $k = 1$ being obvious. Assume $k \ge 2$ and that the result is true for all pairs (k', d') such that $1 \le k' \le k-1$ and $1 \le d' \le 2k' = 1$. Let $D \subset \mathbb{P}^2$ be a line such that $a := \sharp(S \cap D)$ is maximal. Set $A := S \backslash S \cap D$. Hence $\sharp(A) = d - a$. By assumption we have $a \le k+1$. Hence $h^1(D, \mathcal{I}_{D \cap S, D}(k)) = 0$. A standard exact sequence (the so-called Horace Lemma) show that $h^1(\mathbb{P}^2, \mathcal{I}_S(k)) \le h^1(\mathbb{P}^2, \mathcal{I}_A(k-1))$. Hence it is sufficient to prove $h^1(\mathbb{P}^2, \mathcal{I}_A(k-1)) = 0$. Since we may assume $d \ge 2$, we also have $a \ge 2$. Hence $\sharp(A) = d - a \le 2(k-1)+1$. Let $R \subset \mathbb{P}^2$ be a line such that $b := \sharp(A \cap R)$ is maximal. If $b \le k$, then we may apply the inductive assumption. Assume $b \ge k+1$. Since $a \ge b$ and $A \cap D = \emptyset$, we get $d \ge a + b \ge 2k+2$, contradiction. □

Lemma 2. *Fix a finite subset $S\mathbb{P}^2$ and an integer $k > 0$. Set $d := \sharp(S)$. If there is no line $D \subset \mathbb{P}^2$ such that $\sharp(S \cap D) \ge k+2$, then $h^0(\mathbb{P}^2, \mathcal{I}_S(k)) \le (k+2)(k+1)/2 - \min\{d, 2k+1\}$.*

Proof. If $d \le 2k+1$, then apply Lemma 1. If $d \ge 2k+2$, then take any $S' \subset S$ such that $\sharp(S') = 2k+1$ and apply Lemma 1 to S'. □

Lemma 3. *Fix a finite subset $S\mathbb{P}^2$ and an integer $k > 0$. Set $d := \sharp(S)$. Assume the existence of at least two lines line $D_1, D_2 \subset \mathbb{P}^2$ such that $\sharp(S \cap D_i) \ge k+2$ for all i. Then $h^0(\mathbb{P}^2, \mathcal{I}_S(k)) \le (k=2)(k+1)/2 - \min\{d, 2k+1\}$.*

Proof. The set $S \cap D_1$ imposes $k+1$ independent conditions to $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$. The set $S \cap D_2 \backslash S \cap D_1 \cap D_2$ imposes k independent conditions to $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-1))$. Use Horace Lemma. □

Proposition 2. *Let $Y \subset \mathbb{P}^3$ be a reduced curve such that each of its irreducible components are non-degenerate. Set $d := \deg(Y)$. Then $h^1(Y, \mathcal{O}_Y(t)) = 0$ for all $t \geq m := \lfloor d/2 \rfloor + 1$. We have $h^1(Y, \mathcal{O}_Y(m-1)) > 0$ if and only if d is even and Y is the complete intersection of an integral quadric surface and a degree $d/2$ surface.*

Proof. Fix a general plane $H \subset \mathbb{P}^3$. Hence $Y \cap H$ is a reduced set. For every integer t we have an exact sequence of coherent sheaves on $\mathbb{P}^3$:

$$0 \to \mathcal{I}_Y(t-1) \to \mathcal{I}_Y(t) \to \mathcal{I}_{Y \cap H, H}(t) \to 0 \,. \tag{1}$$

Since $\dim(Y) = 1$, a twist of the ideal sheaf sequence of Y gives $h^2(\mathbb{P}^3, \mathcal{I}_Y(z)) = h^1(Y, \mathcal{O}_Y(z))$ for every $z \in \mathbb{Z}$. Since $\mathcal{O}_Y(1)$ is very ample, the sequence of non-negative integers $\{h^1(Y, \mathcal{O}_Y(t))\}_{t \in \mathbb{Z}}$ is non-decreasing and $h^1(Y, \mathcal{O}_Y(t)) = 0$ for all $t \gg 0$. Hence (1) gives the first assertion if $h^1(H, \mathcal{I}_{Y \cap H}(m-1)) = 0$. Lemma 1 gives the last vanishing unless there is a line $D \subset H$ such that $\sharp(D \cap (H \cap Y)) \geq m+1$. If D is exists, then $H \cap Y$ is not in linearly general position. Hence one of the cases (i), (ii), (iii) listed in Remark 1 occurs. In each of these cases the line D is not unique. Apply Lemma 3. The "if" part of the last assertion is obvious and in this case we have $\omega_Y \cong \mathcal{O}_Y(d/2-2)$ and hence $h^1(Y, \mathcal{O}_Y(m-1)) = 1$ (use duality and that a complete intersection is connected). □

Proof of Proposition 1. Use the proof of Proposition 2. □

Proof of Theorem 1. Since the cases $d \leq 4$ are obvious, we may assume $d \geq 5$. The "if " part of the last assertion is obvious by the structure of curves on integral quadric surfaces. By the case $t = 0$ of Proposition 1 we may assume $p_a(Y) =$. The structure of curves on integral quadric surfaces show that it is sufficient to prove that Y is contained in an integral quadric surface. Since every irreducible component of Y spans $\mathbb{P}^3$, Y is contained in no reducible quadric. Hence it is sufficient to prove $h^0(\mathbb{P}^3, \mathcal{I}_Y(2)) > 0$.

(a) Here we assume that $H \cap Y$ is contained in an irreducible conic D. Castelnuovo' method ([2], Cor. 3.2) and the cohomology of the ideal sheaf of any finite subset of D (seen as a subset of H) gives $p_a(Y) \leq \lfloor d/2 \rfloor \lceil d/2 \rceil - d + 1$. Since each component of Y is non-degenerate, any quadric surface containing Y is irreducible. Hence it is sufficient to prove that Y is contained in a quadric surface. Since $d \geq 5$, the smooth conic $D \subset H$ containing $Y \cap H$ is unique. Fix another general plane H' and let D' be the only smooth conic of H' containing $Y \cap H'$. Fix any $P \in Y \backslash (Y \cap (H \cup H')$. Using homogeneous coordinates $x_0, \dots, x_3$ such that $P = (1;1;0;0)$, $H = \{x_0 = 0\}$ and $H' = \{x_1 = 0\}$ it is easy to check that there is a unique quadric surface S containg P and such that $S \cap H = D$ and $S \cap H' = D'$. Since $\sharp(S \cap Y) \geq 2d+1$, S contains at least one irreducible component of Y. More precisely, Bezout theorem gives that S contains at least the irreducible component Y_h containing P. Fix $j \in \{1, \dots, s\}$ such that $j \neq h$. If $Y_j \cap Y_h$ contains a point outside $H \cup H'$, then Bezout gives $Y_j \subset S$. For general H, H' we may assume $H \cap H' \cap Y \subset Y_{reg}$. Hence either $Y_j \cap Y_h = \emptyset$ or $Y_j \subset S$. Since Y is connected, in finitely many steps we get that every irreducible component of Y is contained in S. Hence $Y \subset S$, concluding the proof in this case.

(b) Here we assume $h^0(H, \mathcal{I}_{Y\cap H,H}(2)) = 0$. If $Y \cap H$ is in linearly general position apply Lemma 1 to the finite set $Y \cap H$. In the general cases apply also lemmas 2 and 3 to the finite set $Y \cap H$. Since $h^0(H, \mathcal{I}_{Y\cap H,H}(2)) = 0$, in the case $k = 2$ we get strict inequality. Hence $p_a(Y) < \lfloor d/2 \rfloor \lceil d/2 \rceil - d + 1$ (proof of Proposition 2), contradiction.

(c) By parts (a) and (b) we may assume that $H \cap Y$ is contained in an reducible conic $D_1 \cup D_2$. Since $d \geq 5$, $Y \cap H$ is not in linearly general position. Hence some of the irreducible components of Y are as described in Remark 1. In each of these cases we see that a general hyperplane section is contained in no reducible conic, contradiction. □

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