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STABLE SPACE CURVES WITH COOMOLOGICALLY NICE NORMAL BUNDLE

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Abstract: Here we construct many stable reducible curves $X \subset \mathbb{P}^3$ such that $h^1(X, N_X) = 0$ (and sometimes $h^1(X; N_X(-1)) = 0$), where N_X is the normal bundle of X in $\mathbb{P}^3$.

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1. Introduction

Let $\mathbb{K}$ be an algebraically closed base field such that $\mathrm{char}(\mathbb{K}) = 0$. For any coherent sheaf F on $\mathbb{P}^3$ and every integer $i \geq 0$ set $H^i(F) := H^i(\mathbb{P}^3, F)$ and $H^i(F) := H^i(\mathbb{P}^3, F)$. For any reduced pure-dimensional projective curve $Y \subset \mathbb{P}^3$ let $s(Y)$ denote the minimal integer t such that $h^0(\mathcal{I}_Y(t)) \neq 0$. Let $\sigma(Y)$ denote the minimal integer t such that $h^0(H, \mathcal{I}_{Y\cap H}(t)) \neq 0$. Obviously, $s(Y) \geq \sigma(Y)$. For every integer $t \geq 0$ set $h_Y(t) := \binom{t+3}{3} - h^0(\mathcal{I}_Y(t))$, i.e. let h_Y denote the Hilbert function of Y. For every integer $t \geq 0$ set $h_{Y\cap H}(t) := \binom{t+2}{2} - h^0(H, \mathcal{I}_{Y\cap H}(t))$, where $H \subset \mathbb{P}^3$ is a general plane, i.e. let $h_{Y\cap H}$ denote the Hilbert function of the general hyperplane section of Y. We assume that Y is connected, but that it has $x \geq 2$ irreducible components. We assume that every singular point of Y lying in at least two irreducible components of Y is an ordinary node of Y. We fix an ordering $Y_1, \dots, Y_x$ of the irreducible components of Y and set $d_i := \deg(Y_i)$. The x-ple of positive integer $(d_1, \dots, d_x)$ is the multidegree of the x-ple $(Y_1, \dots, Y_x)$ or the multidegree of Y with respect

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to the given ordering of its irreducible components. For all $i, j \in \{1, \ldots, x\}$ such that $i \neq j$ set $k_{ij} = kji = \sharp_i^Y \cap Y_j)$ and $S_{ij} := Y_i \cap Y_j \subset Y_i$. Obviously, $S_{ij} = S_{ji}$ as sets, but the first index i is used to see it as a subset of Y_i. Let $A(Y; Y_1, \ldots, Y_x; d_1, \ldots, d_x)$ be the set of all embeddings of Y into $\mathbb{P}^3$ with multidegree $(d_1, \ldots, d_x)$ with respect to the given ordering of its irreducible components. For every integer $i \in \{1, \ldots, x\}$ set $A_i := Y_1 \cup \cdots \cup Y_i$. We say that Y is *loosely connected* with respect to the ordering $Y_1, \ldots, Y_s$ of its irreducible components (or just loosely connected)if $\sharp(A_i \cap Y_{i+1}) \leq 5$ for every $i \in \{2, \ldots, x\}$ (i.e. if $\sum_{1 \leq j < i} k_{ij} \leq 5$ for every $i \in \{2, \ldots, x\}$) and the each set $A_i \cap Y_{i+1}$ is in linearly general position. Let $B(Y; Y_1, \ldots, Y_x; d_1, \ldots, d_x)$ the open subset of $A(Y; Y_1, \ldots, Y_x; d_1, \ldots, d_x)$ formed by the loosely connected embeddings. Let $E(Y_1, \ldots, Y_x; k_{ij}, 1 \leq i < j \leq x; d_1, \ldots, d_x)$ be the set of all embeddings $W \hookrightarrow \mathbb{P}^3$ of reduced projective curves W satisfying the following conditions:

(a) each singular point of W lying on at least 2 irreducible components of W are ordinary nodes of W;

(b) W has and ordering $W_1, \ldots, W_x$ of its irreducible components such that $W_i \cong Y_i$ (as an abstract curve) for all i;

(c) $\sharp(W_i \cap W_j) = k_{ij}$ for all $i \neq j$;

(d) the embedding has multidegree $(d_1, \ldots, d_x)$ with respect to the ordering $W_1, \ldots, W_x$.

$E(Y_1, \ldots, Y_x; k_{ij}, 1 \leq i < j \leq x; d_1, \ldots, d_x)$ is parametrized by a reduced quasi-projective scheme. Let $F(Y_1, \ldots, Y_x; k_{ij}, 1 \leq i < j \leq x; d_1, \ldots, d_x)$ (resp. $G(Y_1, \ldots, Y_x; k_{ij}, 1 \leq i < j \leq x; d_1, \ldots, d_x)$) denote the open subset of $E(Y_1, \ldots, Y_x; k_{ij}, 1 \leq i < j \leq x; d_1, \ldots, d_x)$ parametrizing the loosely connected (resp. very loosely connected) embeddings. For any locally complete intersection curve $T \subset \mathbb{P}^3$ let N_T denote its normal bundle.

Theorem 1. *Fix integers $x \geq 2$, $d_i \geq 5$, $1 \leq i \leq x$, and $g_i \geq 0$, $1 \leq i \leq x$. Assume that for every $i \in \{1, \ldots, x\}$ there is a smooth and connected curve $C_i \subset \mathbb{P}^3$ such that $\deg(C_i) = d_i$, $p_a(C_i) = g_i$ and $h^1(C_i, N_{C_i}(-2)) = 0$. For all $i, j \in \{1, \ldots, x\}$ fix non-negative integers k_{ij} such that $k_{ij} = k_{ij}$ for all $i \neq j$ and $\sum_{1 \leq j < i} k_{ij} \leq 3$ for all $i \geq 2$. Let Y_i, $1 \leq i \leq x$, be a general smooth curve of genus g_i, with the convention that Y_i is an arbitrary elliptic curve if $g_i = 1$. Then there exists $W \in G(Y_1, \ldots, Y_x; k_{ij}, 1 \leq i < j \leq x; d_1, \ldots, d_x)$ such that*

(a) $h^1(W, N_W(-1)) = 0$;

(b) $h_{W \cap H}(t) = \max\{0, \binom{t+2}{2} - d_1 - \cdots - d_x\}$.

See [3], [4] and the two unpushished preprints of C. Walter quoted for a list of all (resp. almost all) almost all pairs (d_i, g_i) such that there is a smooth and connected curve $C_i \subset \mathbb{P}^3$ such that $\deg(C_i) = d_i$, $p_a(C_i) = g_i$ and $h^1(C_i, N_{C_i}(-1)) = 0$ (resp. $h^1(C_i, N_{C_i}(-2)) = 0$). For highly connected reducible curves we only point out the following result known long time ago and used in unpublished joint work by P. Ellia and myself.

Theorem 2. *Fix integers $d > 0$, $k \geq 1$, $s_i \geq 2$, $1 \leq i \leq k$, a_i, $1 \leq i \leq k$, such that $1 \leq a_i \leq s_i(s_i+1)/2$ for all i and $(d-2)(d-3)/2 \leq \sum_{i=1}^{k} a_i \leq (d^2+3d)/2$. Then there exists a connected nodal curve $X \subset \mathbb{P}^3$ with arithmetic genus $-k + \sum_{i=1}^{k}(a_i + 1 + s_i(s_i^2 + 3s_i + 1)/6 - s_i(s_i+2) + (d-1)(d-2)/2$, degree $d + \sum_{i=1}^{k} s_i(s_i+1)/2$, $k+1$ irreducible components, say X_i, $0 \leq i \leq k$, each X_i is smooth, $\deg(X_i) = s_i(s_i+1)/2$ and $p_a(X_i) = 1 + s_i(s_i^2+3s_i+1)/6 - s_i(s_i+2)$ for all $1 \leq i \leq k$, $X_i \cap X_j = \emptyset$ if $1 \leq i < j \leq k$, $\deg(X_0) = d$, $p_a(X_0) = (d-1)(d-2)/2$, and $\sharp(X_0 \cap X_i) = a_i$ for every $i \in \{1, \dots, k\}$. If $\sum_{i=1}^{k} a_i \geq (d-1)(d-2)/2$, then we also have $h^1(X, N_X(-1)) = 0$.*

Since the curve C_s described in Example 1 below satisfies $h^1(C_s, N_{C_s}(-2)) = 0$, the interested reader may easily extend the "$h^1(X, N_X) = 0$" part of Theorem 2 taking instead of d a pairs of positive integers d_1, d_2 and instead of X_0 a smooth curve of bidegree (d_1, d_2) contained a smooth quadric surface.

Remark 1. The interested reader may use [7], Lemma 2.1 (applied several times) and its proof to get several reducible and nodal space curves with coomologically nice restricted tangent bundle.

See [4], [1] and [5] for the normal bundle of stick figures.

2. The Proofs

Proof of Theorem 1. Since $h^1(C_i, N_{C_i}(-2)) = 0$, $h^1(C_i, N_{C_i}) = 0$. Hence C_i is a smooth point of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^3)$ of $\mathbb{P}^3$. Let Γ_i denote the unique irreducible component of $\mathrm{Hilb}(\mathbb{P}^3)$ containing C_i. If $g_i \geq 2$ then Γ_i dominates $\mathcal{M}_{g_i}$. If $g_i = 1$, then each smooth elliptic curve has a degree $\deg(C_i)$ embedding (with, say, image D_i) such that $D_i \in \Gamma_i$ and $h^1(D_i, N_{C_i}(-2))$. By the assumptions on Y_i we may take $D_i \cong Y_i$. Hence $F(Y_1, \dots, Y_x; k_{ij}, 1 \leq i < j \leq x; d_1, \dots, d_x) = F(D_1, \dots, D_x; k_{ij}, 1 \leq i < j \leq x; d_1, \dots, d_x)$. For all $i, j \in \{1, \dots, x\}$ fix a general $A_{ij} \subset D_i$ such that $\sharp(S_{ij}) = k_{ij}$. Since D_i is nondegenerate, the set $\cup_{j \neq i} S_{ij}$ is in linearly general position. Set $A_1 := W_1 := D_1$. Since $\sharp(A_{12}) = \sharp(A_{21}) = k_{12} \leq 5$ and the sets A_{12} and A_{21} are in linearly general position, there is $h \in \mathrm{Aut}(\mathbb{P}^3)$ such that $h(A_{21}) = A_{12}$. Set $W_2 := h(D_2)$ and $A_2 := A_1 \cup W_2$. Moving $D_2 \in \Gamma_2$ and the set A_{21} we may also assume $W_1 \cap W_2 = A_{12}$ and that each point of $W_1 \cap W_2$ is an ordinary node of A_2. If $x = 2$, then set $W := A_2$. Now assume $x \geq 3$. For general A_{13} and A_{23} we may also assume that the set $A_{13} \cup h(A_{23})$ is in linearly general position. Since $k_{13} + k_{23} \leq 5$, there is $h_2 \in \mathrm{Aut}(\mathbb{P}^3)$ such that $h_2(A_{31}) = A_{13}$ and $h_2(A_{32}) = h(A_{23})$. Set $W_3 := h_2(D_3)$ and $A_3 := A_2 \cup W_3$. Moving $D_3 \in \Gamma_3$ and the set $A_{31} \cup A_{32}$ we may also assume $W_1 \cap W_3 = A_{13}$, $W_2 \cap W_3 = h(A_{23})$ and that each point of $A_2 \cap W_3$ is an ordinary node of A_3. If $x = 3$, then set $W := A_3$. If $x \geq 4$ we continue in the same way and get $W \in F(Y_1, \dots, Y_x; k_{ij}, 1 \leq i < j \leq x; d_1, \dots, d_x)$. Thus $W \in G(Y_1, \dots, Y_x; k_{ij}, 1 \leq i < j \leq x; d_1, \dots, d_x)$. Now we prove part (i) by induction on x. Since $A_1 = W_1 = D_1$ we have $h^1(A_1, N_{A_1}(-2)) = 0$ and in particular $h^1(A_1, N_{A_1}(-1)) = 0$. Assume $h^1(A_i, N_{A_i}(-1)) = 0$ for some $i \in \{1, \dots, x-1\}$. To conclude the inductive step it is sufficient to prove $h^1(A_{i+1}, N_{A_{i+1}}(-1)) = 0$. Look at [6]. We have $N_{A_{i+1}}|A_i \cong N^+_{A_i}$, where the latter rank two vector bundle on A_i is obtained from N_{A_i} making $\sum_{1 \leq i \leq j \leq i} k_{ji+1}$ positive elementary transformations, eah of them supported by a different point of $A_i \cap W_{i+1}$ and determined by the tangent line of W_{i+1} at the corresponding point ([6]). We have $N_{A_{i+1}}|W_{i+1} \cong N^+_{W_{i+1}}$, where the latter rank 2 vector bundle on W_{i+1} has a similar description. Hence we have the following Mayer-Vietoris exact sequence

$$0 \to N_{A_{i+1}}(-1) \to N_{A_i}(-1)^+ \oplus N^+_{W_{i+1}}(-1) \to N_{A_{i+1}}|A_i \cap W_{i+1} \to 0\,. \qquad (1)$$

Since $N^+_{A_i}$ is isomorphic to a rank 2 subsheaf of N_{A_i} and $h^1(A_i, N_{A_i}(-1)) = 0$, we have $h^1(A_i, N^+_{A_i}(-1)) = 0$. Similarly, we have $h^1(W_{i+1}, N^+_{W_{i+1}}(-2)) = 0$. Since W is very loosely connected, $A_i \cap W_{i+1}$ is contained in a plane. Hence the vanishing of $h^1(W_{i+1}, N_{W_{i+1}}(-2))$ implies $h^1(W_{i+1}, N_{W_{i+1}}(-1 - A_i \cap W_{i+1})) = 0$. Hence $h^1(W_{i+1}, N^+_{W_{i+1}}(-1 - a_i \cap W_{i+1})) = 0$. Hence the restriction map $H^0(W_{i+1}, N^+_{W_{i+1}}(-1)) \to H^0(A_i \cap W_{i+1}, N_{A_{i+1}}(-1)|A_i \cap W_{i+1})$. Hence (1) gives part (i). Part (ii) is a consequence of part (i) and [8], Theorem 1.5. □

Example 1. Here we follow [2], §1. Fix an integers $s \geq 2$ and let $C_s \subset \mathbb{P}^3$ be any smooth curve with the following minimal free resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-s-1)^{\oplus s} \to \mathcal{O}_{\mathbb{P}^3}(-s)^{\oplus(s+1)} \to \mathcal{I}_{C_s} \to 0\,. \qquad (2)$$

C_s exists, it is irreducible, $\deg(C_s) = s(s+1)/2$, and $p_a(C_s) = 1 + s(s^2+3s+1)/6 - s(s+1)$. We have $h^1(C_s, N_{C_s}(-2)) = 0$ ([2], proof of Proposition 2).

Proof of Theorem 2. Fix a plane $H \subset \mathbb{P}^3$. Let $X_i := C_{s_i}$, $1 \leq i \leq k$, be general curves described in Example 1. Since each the k-ple $(C_{s_1}, \dots, C_{s_k})$ is general, we may assume $C_{s_i} \cap C_{s_j} = \emptyset$ for all $1 \leq i < j \leq k$. Set $Y :=$

$\cup_{i=1}^{k} C_{s_i}$. Since $h^1(C_{s_i}, N_{C_{s_i}}(-2)) = 0$ and $C_{s_i} \cap C_{s_j} = \emptyset$ for all $1 \le i < j \le k$, we have $h^1(Y, N_Y(-2)) = 0$. Hence $h^1(Y, N_Y(-1)) = 0$. Hence for general $(C_{s_1}, \dots, C_{s_k})$ we may assume that $Y \cap H$ is a general union of $\deg(Y)$ points. Fix $A_i \subseteq C_{s_i} \cap H$ such that $\sharp(A_i) = a_i$. This is possible, because $a_i \le s_i(s_i + 1)/2 = \deg(C_{s_i})$. Let X_0 be a general degree d curve. For a general X_0 we may asume that $S \subset X_0$, where S is a prescribed subset of H formes by at most $(d^2 + 3d)/2$ general points of H. Since $\sum_{i=1}^{k} a_i \le (d^2 + 3d)/2$, we may assume $\cup_{i=1}^{k} A_i \subset X_0$. For general Y and X_0 we may assume $Y \cap X_0 = \cup_{i=1}^{k} A_i$. Set $X := Y \cup X_0$. $a_i > 0$ for all i, the curve X is connected. Since $\sum_{0 \le i < j \le k} \sharp(X_i \cap X_j) = \sum_{i=1}^{k} a_k$, we have $p_a(X) = \sum_{i=0}^{k} p_a(X_i) + \sum_{i=1}^{k} a_i - k$. Hence it is sufficient to prove $h^1(X, N_X) = 0$ (resp. $h^1(X, N_X(-1)) = 0$ if $\sum_{i=1}^{k} a_i \ge (d-2)(d-1)/2$. We follow the main constructions of [6]. We have $N_{X_0} \cong \mathcal{O}_{X_0}(1) \oplus \mathcal{O}_{X_0}(d)$. Hence $h^1(X_0, N_{X_0}) = h^0(X_0, \mathcal{O}_{X_0}(d-4)) = \max\{0, (d-2)(d-1)/2\}$. Hence for any integer $c \ge \max\{0, (d-2)(d-1)/2\}$ the general vector bundle F obtained from N_{X_0} making c general positive elementary transformations satisfies $h^1(X_0, F) = 0$. If $c \ge (d-2)(d-1)/2$, then $h^1(X_0, F(-1)) = 0$. Fix a general $S \subset H$ such that $\sharp(S) = \deg(Y)$. Let $\{L_P\}_{P \in S}$ be a general set of $\deg(Y)$ lines of $\mathbb{P}^3$ with the only restriction that $P \in L_P$ for every $P \in S$. Since $h^1(Y, N_Y(-2)) = 0$, an obvious generalization of [8], Theorem 1.5, gives that we may take Y as above such that $S \subset Y \cap H$ and for every $P \in S$ the line L_P is the tangent line to Y at P. Hence $N_X|X_0$ is obtained from N_{X_0} making $\sum_{i=1}^{k} a_i$ general poisitive elementary transformations. Since $\sum_{i=1}^{k} a_i \ge \max\{0, (d-2)(d-1)/2\}$, we get $h^1(X_0, N_X|X_0) = 0$. If $\sum_{i=1}^{x} a_x \ge (d-1)(d-2)/2$, then we also get $h^1(X_0, N_X|X_0) = 0$. Consider the Mayer-Vietoris exact sequence

$$0 \to N_X \to N_X|Y \oplus N_X|X_0 \to N_X|Y \cap X_0 \to 0\,. \tag{3}$$

We just saw that $h^1(X_0, N_X|X_0) = 0$ and that $h^1(X_0, N_X|X_0) = 0$ if $\sum_{i=1}^{x} a_x \ge (d-1)(d-2)/2$. Since $N_X|Y$ is obtained from N_Y making some positive elementary transformations, $h^1(Y, N_X(-2)|Y) = 0$ and in particular $h^1(Y, N_X|Y(-1)) = 0$. Since $h^1(Y, N_X(-2)|Y(= 0$, the restriction maps $H^0(Y, N_X(-1)) \to H^0(Y \cap H, N_X(-1)|Y \cap H)$ and $H^0(Y, N_X) \to H^0(Y \cap H, N_X|Y \cap H)$ are surjective. Since $Y \cap H$ is zero-dimensional and $Y \cap X_0 \subseteq Y \cap H$, the restriction maps $H^0(Y \cap H, N_X(-1)|Y \cap H) \to H^0(Y \cap X_0, N_X(-1)|Y \cap X_0)$ and $H^0(Y \cap H, N_X|Y \cap H) \to H^0(Y \cap X_0, N_X|Y \cap X_0)$. Hence all the assertions follow from the long cohomology exact sequence of (3). □

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