International Journal of Pure and Applied Mathematics
Volume 51 No. 3 2009, 431-435

CONTINUITY OF MAPS IN TERMS OF CLUSTER POINTS

A. Goel[1] §, G.L. Garg[2]

[1]Department of Applied Sciences
Punjab Engineering College
Deemed University
Chandigarh, 160 012, INDIA
e-mail: ashagoel30@yahoo.co.in

[2]Department of Mathematics
Punjabi University
Patiala, 147002, INDIA

Abstract: In this paper, among other results, characterizations of continuous maps from k-spaces in terms of cluster points of sequences are obtained.

AMS Subject Classification: 54A20, 54C05, 54C10, 54D10, 54D15, 54D20, 54D30, 54D50
Key Words: cluster points, continuous, k-space, compact-preserving, compact, totally sequentially cluster preserving, totally sequentially subcontinuous

1. Introduction

In 1968, Fuller [4] introduced the concept of subcontinuous maps in terms of cluster point of nets and proved the following result.

Theorem 1.1. (see [1], Theorem 3.4) *Let $f : X \to Y$ be any map with closed graph. If f is subcontinuous, then f is continuous.*

In 1988, Piotrowski and Szymanski [5] proved the following result.

Theorem 1.2. (see [4], Theorem 5) *Let $f : X \to Y$ be any map where X is Fréchet and Y is countably compact. If f has closed graph, then f is continuous.*

Received: January 7, 2009

§Correspondence author

In 1997, we gave a generalization of the above result and proved the following theorem.

Theorem 1.3. (see [3], Theorem 4) *Let $f : X \to Y$ be any map where X is Fréchet and Y is B-W compact. If f has closed graph, then f is continuous.*

Also it is known that a map $f : X \to Y$ where X is Fréchet is continuous if and only if $\{x_n\} \to x$ implies $f(x_n) \to f(x)$ for each point x of X. In this paper, we introduce the concept of totally sequentially subcontinuous and totally sequentially cluster preserving maps and obtain characterizations of continuous maps in terms of cluster points of sequences from k-spaces instead of Fréchet spaces. Moreover, our Proposition 3.2 is an analogue of our Theorem 1, see [3], for continuity of compact maps.

2. Preliminaries

Throughout this paper, the term space means a general topological space. No separation axioms are assumed (a normal space is not assumed to be T_1) and no map is assumed to be continuous or onto unless mentioned explicitly. The notation cl(A) stands for the closure of the subset A in the space X. Besides, the sequences are denoted by the symbols $\{x_n\}$ and their convergence to a point x is then denoted by $\{x_n\} \to x$. A point x in X is said to be a *cluster point* of a subset A of X if every neighbourhood of x contains an infinite number of points of A.

A space X is said to be:

(i) *Fréchet space* (*closure sequential* in the terminology of Wilansky [1]) if for each subset A of X, $x \in cl(A)$ implies there exists a sequence $\{x_n\}$ in A which converges to x.

(ii) *k-space* if O is open (equivalently: closed) in X whenever $O \cap K$ is open (closed) in K for every compact subset K of X.

(iii) *KC space* if each compact set is closed.

(iv) *S_2-space* (see Garg and Goel [3]) if for every pair of points x and y in X, whenever one of these has a neighbourhood not containing the other, then x and y have disjoint neighbourhoods.

For arbitrary topological spaces X and Y, a map $f : X \to Y$ will be said to be:

(i) *compact-preserving (compact)* (see Garg and Goel [3]) if image (inverse image) of each compact set is compact.

(ii) *perfect* (see Garg and Goel [3]) if it is continuous, closed and has compact fibers $f^{-1}(y)$, for each point $y \in Y$.

(iii) *totally sequentially cluster preserving* if whenever a sequence $\{x_n\}$ has x as a cluster point in X, the sequence $\{f(x_n)\}$ has $f(x)$ as a cluster point in Y.

(iv) *totally sequentially subcontinuous* if whenever a sequence $\{x_n\}$ has a cluster point in X, the sequence $\{f(x_n)\}$ has a cluster point in Y.

Remark 2.1. It is easy to see that every Fréchet space is a k-space and a T_2-space or a regular space is S_2.

Remark 2.2. Obviously, every totally sequentially cluster preserving map is totally sequentially subcontinuous.

3. Main Results

Theorem 3.1. *Let $f : X \to Y$ be any map where X is a k-space which is either KC or S_2 or normal and Y is KC space which is either hereditarily Lindelof or hereditarily paracompact. Then f is continuous if and only if it is totally sequentially cluster preserving with closed fibers.*

Proof. First suppose f is continuous. Let $\{x_n\}$ be any sequence in X which clusters at x and V be any neighbourhood of $f(x)$ in Y. Then there exists a neighbourhood U of x in X such that $f(U)$ is contained in V. Now $\{x_n\}$ clusters at x in X implies U contains an infinite number of terms of $\{x_n\}$. Therefore, V contains an infinite number of terms of $\{f(x_n)\}$ and hence $\{f(x_n)\}$ clusters at $f(x)$ in Y, implying thereby that f is totally sequentially cluster preserving. Also f has closed fibers, since Y is KC and f is continuous. Conversely, to prove continuity of f, it is sufficient to prove that f is compact preserving in view of our Proposition 3.1 given below. Let K be a compact subset of X and $\{f(x_n)\}$ be any sequence in the set $f(K)$. Now $\{x_n\}$, being a sequence in the compact set K, has a cluster point x in K. Then, by hypothesis, $f(x)$ is a cluster point of $\{f(x_n)\}$ in $f(K)$ and so $f(K)$ is countably compact. Since Y is either hereditarily Lindelof or hereditarily paracompact, it follows that $f(K)$ is compact and hence f is compact-preserving. □

Proposition 3.1. (see [3], Theorem 2) *Let $f : X \to Y$ be any map, where Y is KC space and X is a k-space which is either KC or S_2 or normal. Then f is continuous if and only if it is compact preserving and has closed fibers.*

The following Theorem 3.2 is a sequential analogue of Theorem 1.1 for closed maps.

Theorem 3.2. *Let $f : X \to Y$ be any closed map, where X is a KC, k-space and Y is either hereditarily Lindelof or hereditarily paracompact. Then f is continuous (perfect) if and only if it is totally sequentially subcontinuous with closed (compact) fibers.*

Proof. Only the non-parenthesis part requires proof. First suppose f is continuous. Then f is totally sequentially cluster preserving as in the proof of Theorem 3.1 and hence f is sequentially subcontinuous by Remark 2.2. Also f has closed fibers since X is T_1 and f is a closed and continuous map. Conversely, suppose f is totally sequentially subcontinuous. To prove continuity of f, it is sufficient to prove that f is compact preserving in view of Proposition 3.2 given below. Let K be a compact subset of X and $\{f(x_n)\}$ be any sequence in the set $f(K)$. Now $\{x_n\}$, being a sequence in the compact set K, has a cluster point x in K. Then, by hypothesis, $\{f(x_n)\}$ has a cluster point (in $f(K)$, since $f(K)$ is closed in Y) and so $f(K)$ is countably compact. Since Y is either hereditarily Lindelof or hereditarily paracompact, it follows that $f(K)$ is compact and hence f is compact-preserving. □

The following proposition is an analogue of our Theorem 1 (see [3]) for continuity of compact maps.

Proposition 3.2. *$f : X \to Y$ be a closed map, where X is a KC, k-space and Y is arbitrary. Then f is continuous (perfect) if and only if it is compact-preserving with closed (compact) fibers.*

Proof. Only the non-parenthesis part requires proof. Since X is a k-space, it is sufficient to prove that the restriction map $f_K = f/K : K \to f(K)$ is continuous for all compact subsets K of X. Since X is KC, the set K is closed in X. Then f is closed implies that the restriction map f_K is closed and since f has closed fibers, it follows that f_K has compact fibers and therefore, f_K is compact by Theorem 3.12 of Fuller [4]. Also f is compact-preserving implies the set $f(K)$ is compact. Now let F be a closed subset of $f(K)$. Then the map f_K is compact implies the set $f_K^{-1}(F)$ is a compact and so closed subset of K, since K is a KC-space. Therefore f_K is continuous and hence f is continuous. □

References

[1] R.V. Fuller, Relations among continuous and various non-continuous functions, *Pacific J. of Math.*, **25** (1968), 495-509.

[2] G.L. Garg, A. Goel, Conditions implying continuity and perfectness of maps, *Acta Math. Hungarica*, **72**, No. 3 (1996), 191-195.

[3] G.L. Garg, A Goel, Perfect maps in compact (countably compact) spaces, *International Journal of Math and Math. Sciences*, Florida, USA, **20**, No. 2 (1997), 405-408.

[4] Z. Piotrowski, A. Szymanski, Closed graph theorem: Topological approach, *Rendiconti del Circolo Matematico di Palermo*, Serie II, **XXXVII** (1988), 88-99.

[5] A. Wilansky, *Topology for Analysis*, Xerox College Publishing, Lexington-Massachusetts-Toronto (1970).