International Journal of Pure and Applied Mathematics
Volume 51 No. 3 2009, 437-440

ON THE POSTULATION OF A GENERAL SUBSET OF A SMOOTH SURFACE X WITH $\dim(\mathrm{Pic}(X)) > 0$

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be a smooth projective surface such that $\dim(\mathrm{Pic}(X)) > 0$. Fix a positive-dimensional family W of line bundles on X and an integer $d > 0$. Let $S \subset X$ be a general union of d points. Here we investigate for which W, d the restriction map $H^0(X, L) \to H^0(S, L|S)$ has maximal rank for all $L \in W$.

AMS Subject Classification: 14K99, 14N05
Key Words: zero-dimensional scheme, Picard group

1. Introduction

Let X be a smooth projective surface such that $\dim(\mathrm{Pic}(X)) > 0$. Fix a positive-dimensional family W of line bundles on X and an integer $d > 0$. Let $S \subset X$ be a general union of d points. Here we investigate for which W, d the restriction map $\rho_{L,S} : H^0(X, L) \to H^0(S, L|S)$ has maximal rank for all $L \in T$. The linear map $\rho_{L,S}$ has maximal rank if and only if $h^0(X, \mathcal{I}_S \otimes L) = \max\{0, h^0(X, L) - d\}$, i.e. if and only if $h^1(X, \mathcal{I}_S \otimes L) = h^1(X, L) + \max\{0, d - h^0(X, L)\}$. We prove the following result.

Theorem 1. *Let X be a smooth projective surface such that $q := \dim(Pic(X)) > 0$. Fix $M \in Pic(X)$ and let W be an algebraic subset of $Pic(X)$ with pure dimension w such that $x := h^0(X, L)$ is constant for all $L \in W$. Assume $x > 0$.*

(a) Fix an integer $d \geq x + w$ and take a general union $S \subset X$ such that

Received: January 15, 2009

$\sharp(S) = d$. If $d \geq x + w + 1$, then $h^0(X, \mathcal{I}_S \otimes L) = 0$ for all $L \in W$. Let $A \subset X$ be a general subset such that $\sharp(A) = x$. If $w > 0$, then there exists $L \in W$ such that $h^0(X, \mathcal{I}_S \otimes L) > 0$, i.e. such that $h^1(X, \mathcal{I}_S \otimes L) > h^1(X, L)$.

(b) Assume $w = 1$ and $h^0(X, \mathcal{I}_A \otimes L) = x - 2$ for every $L \in W$ for a general $A \subset X$ such that $\sharp(A) = x - 2$. Then $h^0(X, \mathcal{I}_S \otimes L) = x - 1$ for every $L \in W$ for a general $S \subset X$ such that $\sharp(S) = x - 1$.

The assumption "$x > 0$" is only made, because if $x = 0$ any finite subset of X has maximal rank with respect to all $L \in W$.

In Section 2 we discuss a specific example (an Abelian surface with $\mathrm{Num}(X) \cong \mathbb{Z}$) and in that case we prove a stronger result (roughly speacking the case $w_j = 2$ of part (b) of the statement of Theorem 1) (see Theorem 2).

2. Proof of Theorem 1 and Particular Example

Let X be an Abelian surface such that $\mathrm{Num}(X) \cong \mathbb{Z}$ and defined over an algebraically closed field. Let $\beta : \mathrm{Num}(X) \to \mathbb{Z}$ be the group isomorphism such that each element of $\beta^{-1}(1)$ is ample. For any integer t set $\mathrm{Pic}^{[t]}(X) := \beta^{-1}(t)\}$. Every connected component of $\mathrm{Pic}^{[t]}(X)$ is a smooth surface isomorphic to the dual Abelian surface $\widehat{A}$.

For any integer $d > 0$ let $\mathrm{Hilb}^d(X)$ denote the Hilbert scheme of all zero-dimensional subschemes of X with length d. Since X is a connected and smooth projective surface, $\mathrm{Hilb}^d(X)$ is a $2d$-dimensional smooth and connected projective variety ([1]. Let $\mathrm{Hilb}^d(X)_=$ denote the dense open subset of $\mathrm{Hilb}^d(X)$ parametrizing the reduced subschemes, i.e. the unions of d distinct points. For any zero-dimensional scheme $Z \subset X$ and all integers $t > 0$, $i > 0$, set $\Sigma(Z, t, i) := \{L \in \mathrm{Pic}^{[t]}(X) : \min\{\dim(\mathrm{Ker}(\rho_{L,Z})), \dim(\mathrm{Coker}(\rho_{L,Z})\} \geq i\}$. Set $\Sigma(Z, t) := \Sigma(Z, t, 1)$.

Remark 1. Let X be an Abelian surface such that $\mathrm{Num}(X) \cong \mathbb{Z}$. For any $M \in \mathrm{Pic}(X)$ let $\beta(M)$ denote the numerical class of M, with the convention that $\beta(M) > 0$ if and only if M is ample and that $\beta : \mathrm{Pic}(X) \to \mathbb{Z}$ is surjective. Hence a line bundle L on X is ample (resp. numerically trivial) if and only if $\beta(L) > 0$ (resp. $\beta(L) = 0$). Obviously, $\beta(L^*) = -\beta(L)$ and $\beta(M \otimes L) = \beta(M) + \beta(L)$ for all M, L. Fix any $R \in \mathrm{Pic}(X)$ such that $\beta(1) = 1$ and set $\alpha := R^2$. Take any $L, M, A \in \mathrm{Pic}(X)$ such that $\beta(L) > 0$, $\beta(M) < 0$ and $\beta(A) = 0$. Serre duality and Kodaira's vanishing (see [2], p. 150, for the case of an ample line bundle on any Abelian variety in arbitrary characteristic) give

$h^1(X,L) = h^1(X,L^*) = h^2(X,L) = 0$. Hence Riemann-Roch gives $h^0(X,L) = \beta(L)^2\alpha/2$. The case $\beta(L) = 1$ gives that α is a positive even integer. Serre duality shows that $h^i(X,M) = 0$ for $i = 0,1$ and $h^2(X,M) = \alpha\beta(M)^2/2$. If $A \cong \mathcal{O}_X$, then $h^0(X,A) = h^2(X,A) = 0$ and $h^1(X,A) = 2$. If A is not trivial, then $h^0(X,A) = h^2(X,A) = 0$. Hence Riemann-Roch gives $h^1(X,A) = 0$ if A is numerically trivial, but not trivial.

Remark 2. Fix positive integers and set $x := t^2\alpha/2$. If $A \subset S$ and $\sharp(S) \le x$, then $\Sigma(A,t,i) \subseteq \Sigma(S,t,i)$. Hence $\Sigma(A,t) \subseteq \Sigma(S,t)$. Conversely, for any $L \notin \Sigma(S,t)$ and any integer $0 \le t < \sharp(S)$ there is $B \subset S$ such that $\sharp(B) = t$ and $L \notin \Sigma(B,t)$.

Theorem 2. *Fix positive integers t, d and set $x := t^2\alpha/2$. Fix a general $S \in Hilb^d(X)_=$.*

(a) If $d = x$, then $\Sigma(S,t)$ is a non-empty 1-dimensional subset of $Pic^{[t]}(X)$.

(b) If $d = x+1$, then $\Sigma(S,t)$ is finite and non-empty.

(c) If $d \ge x+2$, then $\Sigma(S,t) = \emptyset$.

(d) Assume $d = x-1$. Then $\Sigma(S,t)$ is finite and non-empty.

Proof. First assume $d = x$. Fix any finite subset F of $\mathrm{Pic}^{[t]}(X)$. We fixed F and then we take S general. We get $F \cap \Sigma(S,t) = \emptyset$. Since $\mathrm{Pic}^{[t]}(X)$, we get that any irreducible component of $\Sigma(S,t)$ is either a point or a curve. Fix $A \subset S$ such that $\sharp(A) = x-1$ and set $\{P\} := S\backslash A$. If $\Sigma(A,t) \ne \emptyset$, then $\Sigma(S,t) \ne \emptyset$. Assume $\Sigma(A,t) = \emptyset$. Hence for every $L \in \mathrm{Pic}^{[t]}(X)$ there is a unique curve $A_L \in |L|$ such that $A \subset A_L$. Since $A_L \ne A_M$ if $L \ne M$, there is $L \in \mathrm{Pic}^{[t]}(X)$ such that $P \in A_L$, i.e. such that $S \subset A_L$. Since $A \subset A_L$, we get $L \in \Sigma(S,t)$, concluding the proof of part (a).

Now assume $d > x$. Fix $S' \subset S$ such that $\sharp(S') = x$. Since S' may be seen as a general element of $\mathrm{Hilb}^x(X)_=$, we may apply part (a) to S'. We fix S', while we move the points in $S\backslash S'$. We get part (c) and that if $d = x+1$, then $\Sigma(S,t)$ is finite. Assume $d = x+1$. We need to prove that $\Sigma(S,t) \ne \emptyset$. Fix $B \subset S$ such that $\sharp(B) = x$ and set $\{Q\} := S\backslash B$. By part (a) there is a non-empty and one-dimensional $T \subset \mathrm{Pic}^{[t]}(X)$ such that for every $L \in T$ there is $C_L \in |L|$ containing B. Since $\dim(T) > 0$ the union of all curves C_T contains a general point of X. Hence there is $M \in T$ such that $Q \in C_T$. Since $S \subset M$, $M \in B(S,t)$. Hence $B(S,t) \ne \emptyset$. Now assume $d = x-1$. Fix $D \subset S$ such that $\sharp(D) = d-1$ and set $\{O\} := S\backslash D$. If $\Sigma(D,t) \ne \emptyset$, then $\Sigma(S,t) \ne \emptyset$. Assume $\Sigma(A,t) = 0$. Hence for every $L \in \mathrm{Pic}^{[t]}(X)$ there is a 1-dimensional projective space $D_L \subseteq |L|$ such that $C \in |L|$ contains D if and only if $D \in D_L$. Since S is general, D may be seen as a general union of $x-2$ points of X. Let Z_L be the base-locus of D_L. Fix $L \in \mathrm{Pic}^{[t]}(X)$, there is a non-empty $C \in |L|$. Consider the exact sequence

$$0 \to L^* \to \mathcal{O}_X \to \mathcal{O}_C \to 0\,. \tag{1}$$

Kodaira's vanishing (see [2], p. 150, for the case of an ample line bundle on any Abelian variety in arbitrary characteristic) gives $h^1(X, L^*) = 0$. From (1) we get that the map $H^1(X, \mathcal{O}_X) \to H^1(C, \mathcal{O}_C)$ is injective. Hence the restriction map $\eta : \mathrm{Pic}(X) \to \mathrm{Pic}(C)$ has two-dimensional image. Hence the set Z_L determines L, up to finitely many choices. Any $P \notin Z_L$ determines a unique curve $C_{D,L,P} \in D_L$ such that $P \in C_{D,L,P}$. Hence $P \notin A$ and $L \notin \Sigma(A \cup \{P\}, t)$ if and only if $P \notin Z_L$. Thus to prove part (d) it is sufficient to prove that $\cup_{L \in \mathrm{Pic}^{[t]}(X)} Z_L$ is dense in X. Since $\dim(X) = \dim(\mathrm{Pic}(X))$, it is sufficient to prove that $\Sigma(A \cup \{P\}, t)$ is finite for a general $P \in X \backslash A$). This is true, because for any fixed L we have $Z_L = Z_M$ only for finitely many line bundles M. □

Proof of Theorem 1. To prove part (a) adapt the proof of parts (a), (b), (c) of Theorem 2. To prove part (b) adapt the proof of part (d) of Theorem 2. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] J. Fogarty, Algebraic families on an algebraic surface, *Amer. J. Math.*, **90** (1968), 511-521.

[2] D. Mumford, *Abelian Varieties*, Oxford University Press, Bombay (1974).