International Journal of Pure and Applied Mathematics
Volume 51 No. 3 2009, 441-444

ON THE SURJECTIVITY OF THE SYMMETRIC MULTIPLICATION MAP $S^n(H^0(X,E)) \to H^0(X,S^n(E))$ OF A VECTOR BUNDLE E ON A REDUCIBLE CURVE

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Abstract: Here we prove in certain cases the surjectivity of the symmetric multiplication map of the global sections of a vector bundle on a reducible projective curve.

AMS Subject Classification: 14H20, 14H60
Key Words: reducible curve, symmetric multiplication map, vector bundle, general stable vector bundle

1. Introduction

Let $\mathbb{K}$ be an algebraically closed base field such that $\mathrm{char}(\mathbb{K}) = 0$. Fix integers r, d such that $r > 0$ and a smooth and connected projective curve C of genus $q \geq 2$. Let $M(C; r, d)$ denote the set of all stable vector bundles on C with rank r and degree d. Hence the set $M(C; r, d)$ is a non-empty integral variety of dimension $r^2(g-1)+1$. Fix an integer $n > 0$. Take a general $E \in M(C; r, d)$. In [1] we proved that the symmetric power map $S^n(H^0(C,E)) \to H^0(C, S^n(E))$ has maximal rank. Here we consider a similar problem for certain reducible curves. No such clear-cut statement is true on any reducible projective curve (not even for line bundles) for the following reason. Let X be a reducible projective curve and E, F vector bundles on X with "many" sections. Then the multiplication map $H^0(X,E) \otimes H^0(X,F) \to H^0(X, E \otimes F)$ is not injective and the symmetric multiplication map $\sigma_{E,n} : S^n(H^0(X,E)) \to H^0(X, S^n(E))$ is not injective for any integer $n \geq 2$ (it is sufficient the existence of $\sigma \in H^0(X,F)$, $\tau \in H^0(X,E)$

Received: January 15, 2009 © 2009 Academic Publications

and an irreducible component T of X, such that $\sigma \neq 0$, $\tau \neq 0$, $\sigma|\overline{X\backslash T} \equiv 0$ and $\tau|T \equiv 0$). Let X be a reduced projective curve with s irreducible components $X_1, \dots, X_s$, $s \geq 2$. Fix an integer $r \geq 1$ and for every $i \in \{1, \dots, s\}$ a rank r vector bundle E_i on X_i. Let $V(E_1, \dots, E_s)$ be the set of all isomorphism classes of vector bundles E on X such that $E|X_i \cong E_i$ for all i. The set $V(E_1, \dots, E_s)$ is a non-empty and quasi-projective algebraic set. Here we prove the following result.

Theorem 1. *Fix integers $n \geq 2$ and $r \geq 1$. Let X be a reduced, but reducible curve such that each point of X is contained in at most two irreducible components of X and that each irreducible component of X is smooth. Let $X_1, \dots, X_s$, $s \geq 2$, be the irreducible components of X. For each $i \in \{1, \dots, s\}$ set $g_i := p_a(X)$, $Z_i := X_i \cap \overline{X\backslash X_i}$ (scheme-theoretic intersection), $Z := \cup_{i=1}^{s} \cup Z_i$, and $e_i := length(Z_i)$. Assume $g_i \geq 2$ for all i. Fix integers d_i, $1 \leq i \leq s$, such that $\binom{d_i - re_i + r(1-g_i) + n - 1}{n-1} \geq \binom{r+n-1}{n-1}(d_i n/r + 1 - g_i)$ and $d_i - nre_i \geq r(g_i - 1)$ for all i. Fix general $E_i \in M(X_i; r, d_i)$, $1 \leq i \leq s$, and any $E \in V(E_1, \dots, E_s)$. Then the symmetric multiplication map $\sigma_{E,n} : S^n(H^0(X, E)) \to H^0(X, S^n(E))$ is surjective.*

2. The Proof

Proposition 1. *Fix an integer $n \geq 2$ and a reduced projective curve X with two irreducible components, say $X = X_1 \cup X_2$, and a vector bundle E on X. Set $Z := X_1 \cap X_2$ (scheme-theoretic intersection). Set $E_i := E|X_i$. Assume that the symmetric multiplication maps $\sigma_{E_i, Z_i, n} : S^n(H^0(X_i, \mathcal{I}_Z \otimes E_i)) \to H^0(X_i, (\mathcal{I}_Z)^n \otimes S^n(E_i))$ are surjective, $i = 1, 2$. Then the symmetric multiplication map $\sigma_{E,Z,n} : S^n(H^0(X, \mathcal{I}_Z \otimes E)) \to H^0(X, (\mathcal{I}_Z)^n \otimes S^n(E))$ is surjective.*

Proof. Consider the Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to \mathcal{O}_Z \to 0 \,. \tag{1}$$

By twisting (1) with E we get the existence of an isomorphism (induced by the restriction maps) between the vector space $H^0(X, \mathcal{I}_Z \otimes E)$ and the vector space $H^0(X_1, \mathcal{I}_Z \otimes E_1) \oplus H^0(X_2, \mathcal{I}_Z \otimes E_2)$. By twisting (1) with $S^n(E)$ we get the existence of an isomorphism (induced by the restriction maps) between the vector space $H^0(X, \mathcal{I}_Z \otimes S^n(E))$ and the vector space $H^0(X_1, \mathcal{I}_Z \otimes S^n(E_1)) \oplus H^0(X_2, \mathcal{I}_Z \otimes S^n(E_2))$. Seeing $H^0(X, (\mathcal{I}_Z)^n \otimes S^n(E))$ as a subspace of $H^0(X, \mathcal{I}_Z \otimes S^n(E))$ (and similarly for E_1 and E_2) we get an isomorphism (induced by the restriction maps) between the vector space $H^0(X, (\mathcal{I}_Z)^n \otimes S^n(E))$ and the vector space $H^0(X_1, (\mathcal{I}_Z)^n \otimes S^n(E_1)) \oplus H^0(X_2, (\mathcal{I}_Z)^n \otimes S^n(E_2))$. □

Proposition 2. *Fix an integer $n \geq 2$. Let X be a reduced, but reducible projective curve such that every $P \in X$ is coontained in at most two irreducible components of X. Let $X_1, \dots, X_s$, $s \geq 2$, be the irreducible components of X. For each $i \in \{1, \dots, s\}$ set $Z_i := X_i \cap \overline{X \backslash X_i}$ (scheme-theoretic intersection). Set $Z := \cup_{i=1}^{s} \cup Z_i$. Let E be a vector bundle on X. Set $E_i := E|X_i$. Assume that the symmetric multiplication maps $\sigma_{E_i,Z,n} : S^n(H^0(X_i, \mathcal{I}_Z \otimes E_i)) \to H^0(X_i, (\mathcal{I}_Z)^n \otimes S^n(E_i))$ are surjective for all i. Then the symmetric multiplication map $\sigma_{E,Z,n} : S^n(H^0(X, \mathcal{I}_Z \otimes E)) \to H^0(X, (\mathcal{I}_Z)^n \otimes S^n(E))$ is surjective.*

Proof. The zero-dimensional scheme Z is well-defined, because for each singular point P lying on at least two irreducible components of X, these component are exactly 2, say X_i and X_j with $i \neq j$, and the connected component of Z_i with P has its support is equal to the connected component of Z_i with P has its support. The case $s = 2$ is true y Proposition 1. The general case is done by induction on s. □

Proof of Theorem 1. Let $[n]Z$ denote the closed subscheme of X with $(\mathcal{I}_Z)^n$ as its ideal sheaf. Since X_i is smooth, we may see nZ_i as an effective Cartier divisor of X_i. Since each point of X lies on at most two irreducible components of X, $[n]Z \cap X_i = nZ_i$ for all i. The scheme Z_i is a fixed length e_i effective divisor on X_i. Since E_i is general, we may assume that $E_i(-Z_i)$ is general in $M(X_i; r, d_i - re_i)$. Hence [1] gives the surjectivity of the symmetric multiplication map $\sigma_{E_i,Z_i,n} : S^n(H^0(X_i, \mathcal{I}_Z \otimes E_i)) \to H^0(X_i, (\mathcal{I}_Z)^n \otimes S^n(E_i))$. Hence the symmetric multiplication map $\sigma_{E,Z,n} : S^n(H^0(X, \mathcal{I}_Z \otimes E)) \to H^0(X, (\mathcal{I}_Z)^n \otimes S^n(E))$ is surjective (Proposition 2). Hence to conclude it is sufficient to prove that the image Δ of the symmetric multiplication map $\sigma_{E,Z,n} : S^n(H^0(X, \mathcal{I}_Z \otimes E)) \to H^0(X, (\mathcal{I}_Z)^n \otimes S^n(E))$ maps surjectively onto $H^0([n]Z, S^n(E)|[n]Z)$. Since nZ_i is a fixed 0-dimensional subscheme of X_i with length ne_i and E is general, we may see $E_i(-nZ_i)$ as a general element of $M(X_i; r, d_i - rne_i)$. Since $d_i - rne_i \geq r(g_i - 1)$ and $E_i(-nZ_i)$ is general, $H^1(X_i, E_i(-nZ_i)) = 0$. Hence the restriction map $\rho_i : H^0(X_i, E_i) \to H^0(nZ_i, E_i|nZ_i)$ is surjective. Taking $s - 1$ Mayer-Vietoris exact sequences we get that Δ maps surjectively onto $H^0([n]Z, S^n(E)|[n]Z)$. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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