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SPANNED SUBSHEAVES OF THE DUALIZING SHEAF OF A NODAL CURVE

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Abstract: Let X be a nodal projective curve. Here we study the existence of a spanned subsheaf with pure rank 1 (sometimes locally free) of ω_X.

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1. Introduction

Let X be a stable curve of genus g. The dualizing sheaf ω_X is spanned if and only if X has no disconnecting node, i.e. there is no $P \in \mathrm{Sing}(X)$ such that $X\backslash\{P\}$ is not connected (see [2], Part I of Theorem 1.2, or [3], Part (b) of Theorem 3.3, plus the obvious fact that ω_X is not spanned at each disconnecting node). We start with the following two examples of genus g stable curves X such that there is no locally free spanned subsheaf L of ω_X with $h^0(X,L) \geq 2$ (i.e. $L \neq \mathcal{O}_X$). Of course, since ω_X is locally free, in any such example ω_X is not spanned, i.e. X has a disconnecting node. The first example works for any genus $g \geq 2$. For the second example we need to assume g large.

Example 1. Let X be a chain of g curves of genus 1, i.e. assume $X = \cup_{i=1}^{g} T_i$, $p_a(T_i) = 1$ for all i, $T_i \cap T_j \neq \emptyset$ if and only if $|i-j| \leq 1$ and $\sharp(T_i \cap T_{i+1}) = 1$ for all $i \in \{1,\dots,g-1\}$. Set $\mathrm{Sing}(X)'' = \{T_i \cap T_{i+1}\}_{1 \leq i \leq g-1}$. Notice that X has $g-1$ disconnecting nodes. TIt is easy to check that the subsheaf F of ω_X spanned by $H^0(X,\omega_X)$ has the property that $\omega_X/F = \oplus_{P\in \mathrm{Sing}(X)''}\mathbb{K}_P$, where $\mathbb{K}_P$ denote the skyscraper sheaf with $h^0(X,\mathbb{K}_P) = 1$ supported by P.

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Hence $\deg(F) = 2g - 2 - \sharp(\mathrm{Sing}(X)'') = g - 1$ and F is not locally free at any point of $\mathrm{Sing}(X)''$. Every spanned subsheaf of ω_X is contained in F. At each $P \in \mathrm{Sing}(X)''$ the germ F_P of F at P is isomorphic to the maximal ideal of the local ring $\mathcal{O}_{X,P}$ ([4], p. 166). Thus for any locally free subsheaf L of F the torsion sheaf F/L must have every point of $\mathrm{Sing}(X)''$ in its support. Thus $\deg(L) \le \deg(F) - (g-1) \le 0$. If L is also spanned, we get $L \cong \mathcal{O}_X$.

Example 2. Let X be a genus g stable curve such that there is an irreducible component $T \cong \mathbb{P}^1$. Set $k := \sharp(X \cap \overline{X \backslash T})$. Notice that $\omega_X|T$ has degree $k-2$. Assume that at least $k-1$ of the points of $X \cap \overline{X \backslash T}$ are disconnecting node of X. Let F be the subsheaf of ω_X spanned by $H^0(X, \omega_X)$ and let L be any locally free subsheaf of F. Since ω_X is not spanned at any disconnecting node, the inclusion map $j : L \hookrightarrow \omega_X$ drop rank at each disconnecting node of X. Hence $\deg(L|T) \le \deg(\omega_X|T) - (k-1) \le -1$. Thus L is not spanned and any section of L vanishes identically on T. In particular $L \ne \mathcal{O}_X$.

Question 1. Is it possible to describe all genus g stable curves X such that there is no locally free spanned subsheaf L of ω_X with $h^0(X, L) \ge 2$?

Let $\Sigma_1(X)$ denote the set of all disconnecting node of a nodal curve X. We summarize our work concerning a single $P \in \Sigma_1(X)$ in the next remark.

Remark 1. Let X be a nodal curve. Fix $P \in \mathrm{Sing}(X)$ and any $M \in \mathrm{Pic}(X)$. Let U be the set of all zero-dimensional schemes $Z \subset X$ such that $Z_{red} = \{P\}$ and $\mathrm{length}(Z) = 2$. Since the Zariski tangent space of X at P has dimension two, $U \cong \mathbb{P}^1$. Fix any $Z \in U$. Since P is an ordinary node of X, Z is a Cartier divisor of X. Hence $\mathcal{I}_Z \otimes M$ is a line bundle. We have $\chi(\mathcal{I}_Z \otimes M) = \chi(M) - 2$, i.e. $\deg(\mathcal{I}_Z \otimes M) = \deg(M) - 2$. Let A be any line bundle contained in ω_X and such that P is in the support of the coherent sheaf ω_X/A. Hence $A \ne \omega_X$. Since P is an ordinary node, the connected component of ω_X/A supported by P must have length ≥ 2. Since $\omega_X \in \mathrm{Pic}(X)$ we also see the existence of $Z \in U$ such that $\mathcal{O}_Z$ is a subsheaf of ω_X/A. Now assume $P \in \Sigma_1(X)$. Let W be the base scheme of $H^0(X, \omega_X)$. Since $P \in \Sigma_1(X)$, $P \in W_{red}$. Let B denote the set of all locally free rank one subsheaves A of X such that ω_X/A has degree 2 and its supported by P. We just saw that $A \in B$ if and only if A is a subsheaf of ω_X and $\omega_X/A \cong \mathcal{O}_Z$ for some $Z \in U$. Set $B_1 := \{A \in B : h^0(X, A) = h^0(X, \omega_X) - 1\}$.

(a) Here we assume that P with its reduced structure is a connected component of W. Fix any $Z \in U$. Set $L := \mathcal{I}_Z \otimes \omega_X$. Since $P \in \Sigma_1(X)$, $h^0(X, L) \ge h^0(X, \omega_X) - 1$. Since W does not contain Z, $h^0(X, L) < h^0(X, \omega_X)$. Hence $h^0(X, L) = \omega_X) - 1$. Varying $Z \in U$ we find a one-dimensional family of line bundles on X which parametrizes B_1.

(b) Assume that W contains the first infinitesimal neighborhood of P in X, i.e. the length 3 closed subscheme of X with $(\mathcal{I}_P)^2$ as its ideal sheaf. We have $h^0(X, A) = h^0(X, \omega_X)$ for every $A \in B$. Hence $B_1 = \emptyset$.

(c) Here we assume that neither (a) nor (b) occur, i.e. we assume that the connected component W' of W supported by P has length at least two and its zariki tangent space has dimension 1. In this case there is a unique $Z' \in U$ such that $Z' \subseteq W$. In this case B_1 is parametrized by $U \backslash \{Z_1\}$.

To solve the Question 1 for non-locally free subsheaves we introduce the following definitions. Let X be a nodal projective curve. We do not assume that X is connected. Let $\mathcal{B}(X)$ denote the set of all irreducible components of X. For any $S \subseteq \mathrm{Sing}(X)$ let $u_S : X_S \to X$ be the partial normalization of X in which we normalize only the points of S. Set $X_0 := X$. Let $\Sigma_1(X)$ denote the set of all disconnecting nodes of X, i.e. the set of all $P \in X$ such that $X \backslash \{P\}$ has more connected components than X. If $\Sigma_1(X) = \emptyset$, then set $\sigma(X) = 0$ and $X_\infty = X$. If $\Sigma_1(X) \neq \emptyset$, then let $u_1 = v_1 : X_1 \to X_0$ be the partial normalization of X_0 in which we only normalize the points of $\Sigma_1(X)$. Set $\Sigma_2(X) = \Sigma_1(X_1)$. Notice that $v_1|\Sigma_2(X)$ is injective and $v_1(\Sigma_2(X)) \cap \Sigma_1(X)) = \emptyset$. We use v_1 to identify $\Sigma_2(X)$ with a subset of $\mathrm{Sing}(X) \backslash \Sigma_1(X)$. We call this set the set of all second order disconnecting nodes of X. If $\Sigma_2(X) = \emptyset$, then set $\sigma(X) := 1$ and $X_\infty := X_1$. If $\Sigma_2(X) \neq \emptyset$, let $v_2 : X_2 \to X_1$ be the partial normalization of X_1 in which we normalize only the points of $\Sigma_2(X)$. Set $\Sigma_3(X) := \Sigma_1(X_2)$ and $u_2 = v_1 \circ v_2$. Since $u_2|\Sigma_2(X)$ is injective, and $u_2(\Sigma_3(X)) \cap (\Sigma_1(X) \cup \Sigma_2(X)) = \emptyset$, we may safely identify $\Sigma_3(X)$ with the set $v_2(\Sigma_3(X)) \subseteq \mathrm{Sing}(X) \backslash (\Sigma_1(X) \cup \Sigma_2(X))$ and call it the set of all third order disconnecting nodes of X. And so on, defining $v_i : X_i \to X_{i-1}$, $u_i : X_i \to X_i$ and $\Sigma_i(X) \subseteq \mathrm{Sing}(X)$ such that $\Sigma_i(X) \cap \Sigma_j(X) = \emptyset$ for all $i \neq j$. We stop after finitely many steps and call $\sigma(X)$ the last integer i such that $\Sigma_i(X) \neq \emptyset$. If no X_i has a connected component isomorphic to $\mathbb{P}^1$, then set $m(X) := \infty$. If there is $i \geq 0$ such that X_i has a connected component isomorphic to $\mathbb{P}^1$ and either $i = 0$ or X_{i-1} has no connected component isomorphic to $\mathbb{P}^1$, then set $m(X) := i$.

Theorem 1. *Let X be a nodal and connected projective curve. There is no spanned subsheaf F of ω_X with pure rank 1 if and only if $m(X) < \infty$.*

Remark 2. Assume $\Sigma_1(X) \neq \emptyset$. Fix $s \in H^0(X, \omega_X)$ and call s' the section of $u_1^*(\omega_X)$ induced by s. There is a natural map $\omega_{X_1} \to u_1^*(\omega_X)$ ([1]). Each point of $\Sigma_1(X)$ lies on two different irreducible components of X and the natural map $\omega_{X_1} \to u_1^*(\omega_X)$ is injective with cokernel isomorphic to a length $2 \cdot \sharp(\Sigma_1(X))$ skyscraper sheaf (for any $P \in \Sigma_1(X)$ it has a one-dimensional stalk at each of the two point of $u_1^{-1}(P)$). Since s vanishes at each point of

$\Sigma_1(X)$, s' vanishes at each point of the support of the cokernel. Hence it induces $s_1 \in H^0(X_1, \omega_{X_1})$. Now assume $s \neq 0$. Since ω_X has no torsion, s is non-zero at the general point of at least one irreducible component T of X. Let T_1 be the unique irreducible component of X_1 such that $u_1(T_1) = T$. Since s' does not vanishes at a general poiunt of T_1, $s' \neq 0$ and $s_1 \neq 0$. Thus the linear map $H^0(X, \omega_X) \to H^0(X_1, \omega_{X_1})$ defined by $s \mapsto s_1$ is injective. Moreover, if $T \in \mathcal{B}(X)$ and $s|T \neq 0$, then $s_1|T_1 \neq 0$. If $\Sigma_2(X) \neq \emptyset$, then we continue in the same way and get an injective linear map $H^0(X, \omega_X) \to H^0(X_2, \omega_{X_2})$. And so on, until we get an injective map $H^0(X, \omega_X) \to H^0(X_{\sigma(X)}, \omega_{X_{\sigma(X)}})$.

Proof of Theorem 1. First assume $m(X) < +\infty$, say $m(X) = i \geq 0$. Assume the existence of a spanned subsheaf F of ω_X with pure rank 1 and $h^0(X, F) > 0$. Since F has pure rank 1, $F' := u_i^*(F)/u_i^*(F)$ is a depth 1 sheaf with pure rank 1. Since F is spanned and the tensor product is a right exact functor, $u_i^*(F)$ is spanned. Hence F' is spanned. Since F is a subsheaf of ω_X and F' has no torsion, we get an injective map $F' \hookrightarrow u_i^*(\omega_X)$. Since F' is spanned, Remark 2 implies that F' is a subsheaf of ω_{X_i}. Since F' is spanned, for every $C \in \mathcal{B}(X_i)$ there is $s \in H^0(X_i, \omega_i)$ such that s does not vanishes at a general point of C. Taking as C a connected component of X_i isomorphic to $\mathbb{P}^1$ we get $h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}) > 0$, contradiction. Now assume $m(X) = \infty$. Notice that each connected component of $X_{\sigma(X)}$ has positive arithmetic genus. Since $X_{\sigma(X)}$ has no disconnecting node, $\omega_{X_{\sigma(X)}}$ is spanned. Notice that $u_{\sigma(X)*}(\omega_{X_{\sigma(X)}})$ is a subsheaf of ω_X with depth 1 and pure rank 1 (use the existence of the map $\omega_{X_{\sigma(X)}} \to u_{\sigma(X)}(\omega_X)$). Let G be the subsheaf of $u_{\sigma(X)*}(\omega_{X_{\sigma(X)}})$ spanned by $H^0(X_{\sigma(X)}, \omega_{X_{\sigma(X)}})$). Since $H^0(X, u_{\sigma(X)*}(\omega_{X_{\sigma(X)}}) = H^0(X_{\sigma(X)}, \omega_{X_{\sigma(X)}}))$, G is not the zero sheaf. Since G is a subsheaf of a depth 1 sheaf, it has depth 1. Since $\omega_{X_{\sigma(X)}}$ is spanned, for each $C \in \mathcal{B}(X_{\sigma(X)})$ there is a section of $\omega_{X_{\sigma(X)}}$ not vanishing at a general point of C. Hence there is a section of G not vanishing at the general point of $u(C)$. Since G is a subsheaf of a line bundle, we get that G has pure rank 1, concluding the proof of the "only if" part of Theorem 1. □

Remark 3. Everything here works verbatim for an arbitrary Gorenstein and reduced projective curve X such that every $P \in \mathrm{Sing}(X)$ lying on at least two irreducible components of X is an ordinary node of X.

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