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THE INVERSE ULTRAHYPERBOLIC BESSEL OPERATOR $(B^{\alpha})^{-1}$ AS A LINEAR COMBINATION OF CAUSAL RIESZ DERIVATIVES

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Abstract: In this paper the inverse causal ultrahyperbolic Bessel operator is expressed as a linear combination of causal Riesz derivative of different orders. This result generalize another one due to Samko (cf. [14]) related to the operator inverse to the Euclidean Bessel potentials.

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1. Introduction

The aim of this article is to describe the inverse causal (anticausal) Bessel potential operators in terms of causal (anticausal) Riesz derivatives.

We deal with causal (anticausal) Riesz and Bessel potentials whose kernels are generalized functions depending on the causal (anticausal) functions $(P \pm i0)^{\lambda}$ introduced by Gelfand. This note is organized in four sections. In Section 2 we review some of the properties of the parametric families of generalized functions $\{G_{\alpha}(P \pm i0, m, n)\}_{\alpha\in\mathbb{C}}$ and $\{H_{\alpha}(P \pm i0, n)\}_{\alpha\in\mathbb{C}}$ which will be extensively used throughout the paper. The first one is the family of causal

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Bessel kernel, and the second the family of the ultrahyperbolic causal Riesz kernel. In Section 3 the ultrahyperbolic causal Bessel potential operator is introduced also the causal Riesz potentials and their inverses operators: the causal Bessel derivative and causal Riesz derivative that are fractional powers of the Klein-Gordon differential operator and of the D'Alembertian, respectively. It is known that this derivatives that work as a left inverse of the causal Bessel and Riesz potentials may be constructed in terms of causal hypersingular integrals, see [4] and [5].

The main result, Theorem 1, given in Section 4, expresses the inverse ultrahyperbolic Bessel operator in terms of causal Riesz derivatives of different orders. This result contains, as a particulas case, another one due to Samko (cf. [13]) in which the connection between the inverse elliptic Bessel operator and the Laplacian operator is presented.

2. Preliminaries

Let $P = P(t)$ be a nondegenerate quadratic form in n variables of the form

$$P = P\,(t) = t_1^2 + \ldots + t_p^2 - t_{p+1}^2 - \ldots - t_{p+q}^2, \tag{2.1}$$

where $p + q = n$, n the dimension of the space $\mathbb{R}^n$, and $t = (t_1, t_2, \ldots, t_n)$ is a point of $\mathbb{R}^n$.

An important contribution to the theory of generalized functions due to Gelfand (cf. [8], p. 274) is $(P + i0)^\lambda$ distributions defined by the following limit

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \left(P \pm i\varepsilon \, |t|^2 \right)^\lambda, \tag{2.2}$$

where $\varepsilon > 0$, $\lambda \in \mathbb{C}$ and $|t|^2 = t_1^2 + \ldots + t_n^2$. This distributions are analytic in λ, everywhere except at $\lambda = -\frac{n}{2} - k$, $k = 0, 1, 2, \ldots$, where they have simple poles (cf. [8], p. 275).

Analogously, the distributions $\left(m^2 + P \pm i0\right)^\lambda$ are defined (cf. [8], p. 289) by

$$\left(m^2 + P \pm i0\right)^\lambda = \lim_{\varepsilon \to 0} \left(m^2 + P \pm i\varepsilon \, |t|^2 \right)^\lambda, \tag{2.3}$$

where m is a real positive number.

Usually the $(P \pm i0)^\lambda$ may be written as a combination of the P_+^λ and P_-^λ

distributions defined by the following expression

$$P_+^\lambda = \begin{cases} P^\lambda & \text{if } P \geq 0\,, \\ 0 & \text{if } P < 0\,, \end{cases} \qquad P_-^\lambda = \begin{cases} 0 & \text{if } P > 0\,, \\ (-P)^\lambda & \text{if } P \leq 0\,, \end{cases} \tag{2.4}$$

then we have

$$(P \pm i0)^\lambda = P_+^\lambda + e^{\pm i\pi\lambda} P_-^\lambda\,. \tag{2.5}$$

Similarly we get

$$\left(m^2 + P \pm i0\right)^\lambda = \left(m^2 + P\right)_+^\lambda + e^{\pm i\pi\lambda}\left(m^2 P\right)_-^\lambda\ , \tag{2.6}$$

where

$$\left(m^2 + P\right)_+^\lambda = \begin{cases} \left(m^2 + P\right)^\lambda & \text{if } \ (m^2 + P) \geq 0\,, \\ 0 & \text{if } \ (m^2 + P) < 0\,, \end{cases}$$

$$\left(m^2 + P\right)_-^\lambda = \begin{cases} 0 & \text{if } \ (m^2 + P) > 0\,, \\ \left(-m^2 - P\right)^\lambda & \text{if } \ (m^2 + P) \leq 0\,. \end{cases}$$

The Fourier transform of a function f is, by definition

$$\mathfrak{F}[f](\xi) = \int_{\mathbb{R}^n} e^{i<t,\xi>} f(t)\, dt\,, \tag{2.7}$$

where $< t, \xi >= \sum_{i=1}^{n} t_i \xi_i$, and for a distribution T is given by the formula

$$(\mathfrak{F}[T], \varphi) = (T, \mathfrak{F}[\varphi]) \quad \text{(cf. [16])}. \tag{2.8}$$

Let α be a complex number. The causal Bessel kernel of order α is given by the distribution defined by the expression

$$G_\alpha\left(P + i0, m, n\right) = \frac{2^{1-\frac{\alpha}{2}} e^{i\frac{\pi}{2}q} \left(m^2\right)^{\frac{1}{2}\left(\frac{n-\alpha}{2}\right)}}{\Gamma\left(\frac{\alpha}{2}\right)} (P + i0)^{\frac{1}{2}\left(\frac{\alpha-n}{2}\right)} K_{\frac{n-\alpha}{2}}\left(\sqrt{m^2\left(P + i0\right)}\right) \tag{2.9}$$

where m is a positive real number, $K_{\frac{n-\alpha}{2}}$ is the modified Bessel function of third kind of order $\frac{n-\alpha}{2}$ and $(P + i0)$ is the generalized function defined by (2.2).

This family of generalized functions $\{G_\alpha\left(P \pm i0, m, n\right)\}_{\alpha\in\mathbb{C}}$ that have been introduced by Trione (cf. [17]), has many important properties like the following that are collected in the next propositions.

Lemma 1. *Let* $G_\alpha\left(P \pm i0, m, n\right)$ *be the Bessel kernel given by* (2.9), *and let*

$$K^l = \left\{\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} + m^2\right\}^l\ , \tag{2.10}$$

be the n dimensional ultrahyperbolic Klein-Gordon operator iterated l-times, $k \geq 1$, $p+q=n$, and $m>0$. Then

$$K^l \left\{G_\alpha \left(P \pm i0, m, n\right)\right\} = G_{\alpha-2l} \left(P \pm i0, m, n\right); \tag{2.11}$$

$$G_{-2l} \left(P \pm i0, m, n\right) = K^l \left\{\delta(x)\right\}; \tag{2.12}$$

and the Fourier transform of $G_\alpha \left(P \pm i0, m, n\right)$ is (cf. [17]):

$$\mathfrak{F}\left[G_\alpha \left(P \pm i0, m, n\right)\right] = \left(m^2 + Q \mp i0\right)^{-\frac{\alpha}{2}}. \tag{2.13}$$

The proof of Lemma 1, and (2.13) appears in [17].

Let now be the following family of generalized functions $H_\alpha(P \pm i0, n)$ defined as

$$H_\alpha(P \pm i0, n) = \frac{e^{\mp \frac{i\pi\alpha}{2}} e^{\pm \frac{i\pi q}{2}} (P \pm i0)^{\frac{\alpha-n}{2}}}{Dn(\alpha)}, \tag{2.14}$$

where α is a complex number $\alpha \neq n+2r$, $r=0,1,2,\ldots,n$ the dimension of the space and $Dn(\alpha)$ is given by

$$Dn(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}. \tag{2.15}$$

This distributions are causal (anticausal) analogues of the elliptic kernel of M. Riesz (cf. [10]) and was introduced by S.E. Trione in [17]. We observe that H_α has simple poles at $\alpha = n+2l$, $l=0,1,2,\ldots$ which are due to the gamma function which appear in the numerator.

Lemma 2. *Let α and β be complex numbers such that α, β, and $\alpha+\beta$ are different from $n+2r$, $r=0,1,\ldots$ and n, the dimension of the space. Then the following formulae are valid*

$$H_0 = \delta(x), \tag{2.16}$$

$$H_\alpha * H_\beta = H_{\alpha+\beta}, \tag{2.17}$$

$$H_\alpha * H_{-2k} = H_{\alpha-2k}, \tag{2.18}$$

$$L^k \left\{H_\alpha\right\} = H_{\alpha-2k} \tag{2.19}$$

$$\text{and} \quad H_{-2k} = L^k\{\delta\}, \tag{2.20}$$

where L^k is the ultrahyperbolic operator iterated k times, $k \geq 1$:

$$L^k = \left\{\frac{\partial^2}{\partial t_1^2} + \ldots + \frac{\partial^2}{\partial t_p^2} - \frac{\partial^2}{\partial t_{p+1}^2} - \ldots - \frac{\partial^2}{\partial t_{p+q}^2}\right\}^k. \tag{2.21}$$

The proof of Lemma 2 is due to Trione (cf. [17], cf. also [20]).

Making use of the properties (2.16) and (2.19) we have

$$L^k \{H_{2k}\} = \delta \,, \tag{2.22}$$

i.e. H_{2k} is an elementary solution of the operator L^k.

3. Previous Results

According Trione (cf. [18]) the *ultrahyperbolic causal Bessel operator of order* α, $\alpha \in \mathbb{C}$ of a function φ belongs to the space S, the Schwartzian space of functions is the operator given by the following convolution

$$B^\alpha \varphi = G_\alpha (P + i0, m, n) * \varphi \,. \tag{3.1}$$

The ultrahyperbolic Bessel operators corresponding to the complex number α with Re$\alpha > 0$ are named *causal Bessel potentials of order* α.

Without difficult we may observe that the Bessel causal potentials defined by (3.1) are a certain kind of generalization of the Bessel potentials introduce by Aronszajn and Smith (cf. [2]), by Caldern (cf. [3]), also to the studied by Rubin (cf. [11], [12]), by Samko. (cf. [13], [14]), Vladimirov (cf. [21]), and by Nogin (cf. [9]).

Resorting to the theorem of interchanging of the Fourier transform for convolution (cf. [16], Theorem XV, p. 268) we have

$$F\left[B^\alpha \varphi\right] = F\left[G_\alpha\right] \cdot F\left[\varphi\right] = \left(m^2 + Q - i0\right)^{-\frac{\alpha}{2}} \cdot F[\varphi] \,.$$

It is know that the inverse operator $(B^\alpha)^{-1}$ has the form of a hypersingular integral in weighted differences (cf [4], [5]). Namely, for $l > \alpha$, $\alpha \neq 1, 3, 5, \dots$ α a positive real number and $\frac{n+\alpha}{2} \neq 0, -1, -2, \dots$, $\frac{n+\alpha}{2} \neq -\frac{n}{2} - k$, $k = 0, 1, 2, \dots$

$$\begin{aligned}(B^\alpha)^{-1} f \\ &= D^\alpha f = \frac{1}{d_{n,l}(\alpha)} \int_{\mathbb{R}^n} (P + i0)^{-\frac{n+\alpha}{2}} \left\{\left(\Delta_t^l f\right)(x, \lambda_\alpha (P + i0))\right\} dt \,, \end{aligned} \tag{3.2}$$

where $\left(\Delta_t^l f\right)(x, \lambda_\alpha (P + i0))$ is the weighted difference of order l of the function f at the point x with interval t and weight $\lambda_\alpha (P + i0)$ defined as:

$$\left(\Delta_t^l f\right)(x, \lambda_\alpha (P + i0)) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k \lambda_\alpha (k (P + i0)) f (x - kt) \tag{3.3}$$

with

$$\lambda_\alpha (P + i0, m, n)$$

$$= \frac{2^{1-\frac{n+\alpha}{2}} e^{i\frac{\pi}{2}q} \left(m^2\right)^{\frac{1}{2}\left(\frac{n+\alpha}{2}\right)}}{\Gamma\left(\frac{n+\alpha}{2}\right)} \left(P+i0\right)^{\frac{n+\alpha}{4}} K_{\frac{n+\alpha}{2}} \left(\sqrt{m^2\left(P+i0\right)}\right) \quad (3.4)$$

and $d_{n,l}(\alpha)$ is some constante given by

$$d_{n,l}(\alpha) = \begin{cases} \dfrac{\pi^{\frac{n}{2}+1} A_l(\alpha)}{2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)\operatorname{sen}\frac{\alpha\pi}{2}} & \text{where } \alpha \neq 2,4,6,\dots, \\ \dfrac{(-1)^{\frac{\alpha}{2}} \pi^{\frac{n}{2}} 2^{1-\alpha}}{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)} \dfrac{d}{d\alpha} A_l(\alpha) & \text{where } \alpha = 2,4,6,\dots. \end{cases} \quad (3.5)$$

$(B^\alpha)^{-1}$ is know as the causal Bessel derivative of order α, and its Fourier transform is

$$F\left[\mathcal{D}^\alpha f\right](\xi) = \left(m^2+Q-i0\right)^{\frac{\alpha}{2}} F\left[f\right](\xi). \quad (3.6)$$

The expression defined by (3.6) is an analogue to the Bessel derivative defined by Rubin [11], p. 1250, formula (1.8).

It will be pointed out some basic properties of the causal Bessel derivative, like the composition formula that follows

$$\mathcal{D}^{\alpha+\beta} f = \mathcal{D}^\alpha \mathcal{D}^\beta f \quad (3.7)$$

for α, $\beta \in \mathbb{R}^+$, and the one given by the following Lemma 3.

Lemma 3. *Let f be a function in S, K^l the Klein-Gordon operator iterated l-times and $\mathcal{D}^\alpha$ the causal Bessel derivative of order α, $\alpha > 0$. Then*

$$\mathcal{D}^\alpha f = \mathcal{D}^{\alpha-2l}\left\{K^l f\right\} = K^l\left\{\mathcal{D}^{\alpha-2l} f\right\} \quad (3.8)$$

The proof of Lemma 3 appears in [7].

As particular case, for $\alpha = 2l$, $l = 1, 2, \dots$ we have

$$\mathcal{D}^{2l} f = K^l f. \quad (3.9)$$

Moreover, we can observe (cf. [18]) that the inverse causal Bessel operator $(B^\alpha)^{-1}$ and then the causal Bessel derivative of order α is, formally, a fractional power of the Klein-Gordon operator

$$(B^\alpha)^{-1} = \mathcal{D}^\alpha = K^{\frac{\alpha}{2}}. \quad (3.10)$$

Let us now observe the causal Riesz potential of order α given by the convolution

$$R^\alpha f = H_\alpha(P+i0,n) * f, \quad (3.11)$$

where $\alpha \in \mathbb{C}$, f is a function of the space S and H_α as (2.14).

The inversion of causal Riesz potentials will be treated as a similar way which was considered for causal Bessel potentials by using hypersingular causals integral in differences (cf. [6]).

Let α be a real number, $\frac{n+\alpha}{2} \neq -\frac{n}{2} - k$, $k = 0, 1, \ldots$, $\alpha < l$, l non negative integer and f a function of the space S the Schwartz class of infinitely differentiable functions on $\mathbb{R}^n$ decreasing at infinity faster than $|x|^{-1}$, and let $(T_l^\alpha f)(x)$ be the operator defined by the formula

$$(T_l^\alpha f)(x) = \int_{\mathbb{R}^n} (P + i0)^{-\frac{n+\alpha}{2}} \left\{ \left(\Delta_t^l f \right)(x) \right\} dt, \tag{3.12}$$

$\alpha > 0$, $l > 0$, where

$$\left(\Delta_t^l f \right)(x) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k f(x - kt).$$

The operator $(T_l^\alpha f)(x)$ must be interpreted in same sense that in (3.9) for causal Bessel operator.

The formula (3.12) defines an operator that will be called "hypersingular causal integral in differences" by analogy with the integral defined by Samko (cf. [15], 1091, formula $(1, 3)$). Its Fourier transform is given by

$$F[T_l^\alpha f](\xi) = d_{n,l}(\alpha)(Q - i0)^{\frac{\alpha}{2}} F[f]. \tag{3.13}$$

The generalized causal Riesz derivative of order α *of a function* $f \in S$ is defined by

$$(\mathcal{D}^\alpha f)(x) = \frac{1}{d_{n,l}(\alpha)} (T_l^\alpha f)(x), \tag{3.14}$$

where α is a real number, l non negative integer, $l > \alpha$ and $\alpha \neq 1, 3, 5, \ldots$, and $d_{n,l}(\alpha)$ is given by (3.5).

Then, from (3.13), we have

$$F[\mathcal{D}^\alpha f](\xi) = (Q - i0)^{\frac{\alpha}{2}} F[f](\xi). \tag{3.15}$$

This generalized causal Riesz derivatives inherits some properties from the Riesz derivatives. In fact, under the assumption for α, $\beta \in \mathbb{R}^+$, $\varphi \in S$ and $\mathcal{D}^\alpha$ is defined by (3.14) (cf. [6]); we have the semigroup property:

$$\mathcal{D}^{\alpha+\beta}\varphi = \mathcal{D}^\alpha \mathcal{D}^\beta \varphi. \tag{3.16}$$

Another important property of the generalized Riesz derivative is the following relation with the ultrahyperbolic differential operator.

Lemma 4. *Let f be a function belongs to S, L^k ultrahyperbolic differential operator iterated k-times, and $\mathcal{D}^\alpha$ the generalized causal Riesz derivative of order α, $\alpha \in \mathbb{R}$, $\alpha \geq 2k$, $k = 1, 2, \ldots$. Then is valid*

$$\mathcal{D}^\alpha f = \mathcal{D}^{\alpha-2k} \left\{ L^k f \right\} = L^k \left\{ \mathcal{D}^{\alpha-2k} f \right\}. \tag{3.17}$$

The proof of Lemma 4 appears in (cf. [6])

4. Main Results

An interesting direct relation between the two kinds of studied operators the inverse Bessel operator and the causal Riesz derivatives, can be expressed as in the following theorem.

Theorem 1. *Let $\alpha = 2k$, $k = 1, 2, \dots,$ be a real positive number. Then the inverse ultrahyperbolic Bessel operator $\left(B^{2k}\right)^{-1}$ may be written as*

$$\left(B^{2k}\right)^{-1} f = \sum_{j=0}^{k} \binom{k}{j} \left(m^2\right)^{k-j} \mathcal{D}^{2j} f.$$

Proof. Let us start by representing the generalized function $\left(m^2 + P \pm i0\right)^{\lambda}$ as a finite linear combination of $(P \pm i0)^{\lambda}$ distribution. Based on the binomial development of the $\left(m^2 + P \pm i0\right)^{k}$ distribution we have

$$\left(m^2 + P + i0\right)^{k} = \sum_{j=0}^{k} \binom{k}{j} \left(m^2\right)^{k-j} (P + i0)^{j} \tag{4.1}$$

and taking into account that for k a positive integer number

$$\left(m^2 + P + i0\right)^{k} = \left(m^2 + P - i0\right)^{k} = \left(m^2 + P\right)^{k} \quad \text{(cf. [17], p. 21)} \tag{4.2}$$

results

$$\left(B^{\alpha}\right)^{-1} = \sum_{j=0}^{\frac{\alpha}{2}} \binom{\frac{\alpha}{2}}{j} \left(m^2\right)^{\frac{\alpha}{2}-j} (\Box)^{j}.$$

In fact, from (3.9) for $\alpha = 2k$ we can write

$$D^{2k} f = \left(m^2 + \Box\right)^{k} f$$

and from the inversion theorem for causal Bessel potentials we know that (cf. [4]) the Bessel derivative is a left inverse and then

$$D^{2k} B^{2k} f = f,$$

or equivalently the causal Bessel derivative of order $2k$ gives an expression of the inverse Bessel operator

$$D^{2k} = \left(B^{2k}\right)^{-1}.$$

From (3.6) we have the Fourier transform of the causal Bessel derivative, and therefore (see $(B^{\alpha})^{-1}$):

$$F\left[\left(B^{2k}\right)^{-1} f\right] = F\left[D^{2k} f\right] = \left(m^2 + Q - i0\right)^k F\left[f\right] = \left(m^2 + Q\right)^k F\left[f\right]. \tag{4.3}$$

The third equality follows from (4.3)

Replacing (4.1) in (4.3) we obtain

$$F\left[\left(B^{2k}\right)^{-1} f\right] = \sum_{j=0}^{k} \binom{k}{j} \left(m^2\right)^{k-j} (Q - i0)^j F\left[f\right]. \tag{4.4}$$

Remembering the expression of the causal Riesz derivative (cf. (3.15))

$$F\left[\mathcal{D}^{\alpha} f\right] = (Q - i0)^{\frac{\alpha}{2}} F\left[f\right], \tag{4.5}$$

and putting (4.5) in (4.4) it result

$$F\left[\left(B^{2k}\right)^{-1} f\right] = \sum_{j=0}^{k} \binom{k}{j} \left(m^2\right)^{k-j} F\left[\mathcal{D}^{2j} f\right].$$

Moreover, taking into account (4.5) for $\alpha = 2j$, $j = 1, 2, \ldots$ from (3.17) we have $F\left[\mathcal{D}^{2j} f\right] = F\left[\Box^j f\right]$, and by the uniqueness of the Fourier transform

$$\mathcal{D}^{2j} f = \Box^j f. \tag{4.6}$$

From (4.4), (4.5) and (4.6) we obtain

$$\left(B^{2k}\right)^{-1} f = \sum_{j=0}^{k} \binom{k}{j} \left(m^2\right)^{k-j} \Box^j f = \sum_{j=0}^{k} \binom{k}{j} \left(m^2\right)^{k-j} \mathcal{D}^{2j} f. \tag{4.7}$$

This result generalize another due to Samko (cf. [15], pp. 548 and 549, cf. also [9], p. 999) related with the elliptic case, where the inverse Bessel operator of order $2k$ is expressed as a linear combination of the Laplacian.

We have the following:

Corollary 1. *If $m = 1$, and $q = 0$, one has*

$$(B^{\alpha})^{-1} f = \sum_{j=0}^{\frac{\alpha}{2}} \binom{\frac{\alpha}{2}}{j} \triangle^j f, \tag{4.8}$$

where $(B^{\alpha})^{-1} f$ is the inverse Bessel operator of order α and $\triangle$ denote the Laplacian operator.

In fact, by observing that if in own results (4.7) $m = 1$ is considered and taking into account that when $q = 0$ the ultrahyperbolic differential operator (2.21) reduces at the Laplacian the corollary follows.

5. Further Remarks

We begin by remarking that it is not difficult to prove the same results as in (4.7) working in the formal mode as well as Riesz and Trione do it (cf. [10], [19]). In fact, let us iterate Klein-Gordon operator k times:

$$\begin{aligned}\left(\square+m^2\right)^k &= \left\{\square\left(1+m^2\square^{-1}\right)\right\}^k=\square^k\left(1+m^2\square^{-1}\right)^k\\ &= \square^k\sum_{j=0}^{k}\binom{k}{j}\left(m^2\square^{-1}\right)^{k-j}=\square^k\sum_{j=0}^{k}\binom{k}{j}m^{2(k-j)}\left(\square^{-1}\right)^{k-j}\\ &= \sum_{j=0}^{k}\binom{k}{j}m^{2(k-j)}\square^j \qquad (5.1)\end{aligned}$$

Then taking into account (3.10) for $\alpha=2k$ we have

$$\left(B^\alpha\right)^{-1}f=D^\alpha f=\sum_{j=0}^{k}\binom{k}{j}m^{2(k-j)}\square^j f=\sum_{j=0}^{k}\binom{k}{j}m^{2(k-j)}\mathcal{D}^{2j}f\,. \qquad (5.2)$$

The last expression (5.2) coincides whit (4.7).

A mayor generality results could be obtained taking into account the formula (I,1;14) from Aguirre (cf. [1]) which expresses

$$\left(m^2+P+i0\right)^\lambda=\sum_{k=0}^{\infty}\frac{\left(m^2\right)^k}{k!}\frac{\Gamma\left(\lambda+1\right)}{\Gamma\left(\lambda-k+1\right)}\left(P+i0\right)^{\lambda-k};$$

valid for $m^2\le P\left(t\right)$ and $\lambda-k\neq-\frac{n}{2}-l$, $l=0,1,2,\ldots$.

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