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FURTHER RESEARCH OF K-OUT-OF-N:G REDUNDANT SYSTEM WITH REPAIR AND MULTIPLE CRITICAL AND NON-CRITICAL ERRORS

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Abstract: We study properties of the C_0-semigroup generated by the operator corresponding to k-out-of-N:G redundant system with repair and multiple critical and non-critical errors. We prove that the C_0-semigroup is a quasi-compact operator, thus we obtain that it converges exponentially to a project operator. For a special case, we deduce asymptotic property of the time-dependent solution of the system.

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1. Introduction

According to [1], the k-out-of-N:G redundant system with repair and multiple critical and non-critical errors can be described by the following system of equations:

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -h_0p_0(t) + b_1p_1(t) + \sum_{j=0}^{M}\int_0^\infty p_{N-k+1+j}(x,t)\mu_{N-k+1+j}(x)\mathrm{d}x, \quad (1)$$

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = a_{i-1}p_{i-1}(t) - h_ip_i(t) + b_{i+1}p_{i+1}(t), \quad i = 1, 2, \cdots, N-k-1, \quad (2)$$

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$$\frac{\mathrm{d}p_{N-k}(t)}{\mathrm{d}t} = a_{N-k-1}p_{N-k-1}(t) - h_{N-k}p_{N-k}(t), \tag{3}$$

$$\frac{\partial p_j(x,t)}{\partial t} + \frac{\partial p_j(x,t)}{\partial x} = -\mu_j(x)p_j(x,t),$$

$$j = N-k+1, \cdots, N-k+1+M, \tag{4}$$

$$p_{N-k+1}(0,t) = a_{N-k}p_{N-k}(t), \tag{5}$$

$$p_{N-k+1+n}(0,t) = \sum_{i=0}^{N-k} d_{i,n}p_i(t), \quad n = 1, 2, \cdots, M, \tag{6}$$

$$p_0(0) = 1, \quad p_i(0) = 0, \quad i = 1, \cdots, N-k,$$

$$p_j(x,0) = 0, \quad j = N-k+1, \cdots, N-k+1+M. \tag{7}$$

Here $h_0 = a_0 + \sum_{j=1}^{M} d_{0,j}$, $h_n = a_n + b_n + \sum_{j=1}^{M} d_{n,j}$, $\quad n = 1, 2, \cdots, N-k$, $(x,t) \in [0,\infty)\times[0,\infty)$; $p_i(t)$ represents the probability that the system is in state i at time $t \quad (i = 0, \cdots, N-k)$, i is number of failed units (active or by any one of the M non-critical errors); $p_j(x,t)$ represents the probability that at time t, the failed system is in failed state j and has an elapsed repair time of x ($j = N-k+1$ means failure of the system, $j = N-k+1+n, n = 1, \cdots, M$ mean failure of the system corresponding to the n-th critical error); a represents constant failure rate of an active unit; b represents constant repair rate of a unit; c_i represent constant failure rate of the i-th non-critical error $(i = 0, \cdots, M)$; $a_i = (i+1)(N-i)(a + \sum_{n=0}^{M} c_n)$; $b_i = \min(i,r)b$, here r is number of repair facilities; $d_{i,j}$ represent constant critical error rate of the system from state i to state j; $(i = 0, \cdots, N-k, \quad j = N-k+1, \cdots, N-k+1+M)$; $\mu_j(x)$ represent repair rate of repair time when system is in state j ($j = N-k+1, \cdots, N-k+1+M$) and has elapsed repair time of x which satisfy

$$\mu_j(x) \geq 0, \quad \int_0^\infty \mu_j(x)\mathrm{d}x = \infty, \quad j = N-k+1, \cdots, N-k+1+M.$$

In [1], Who Kee Chung firstly established the above mathematical model describing the k-out-of-N:G redundant system with repair and multiple critical and non-critical errors by using supplementary variable technique, then studied static solution of the model. In [7], the authors first proved the existence of a unique positive time-dependent solution of the model by using the Hille-Yosida Theorem, Phillips Theorem and Fottorini Theorem, next by discussing the spectral properties of the operator corresponding to the model and using Theorem 14 in [4], they obtained that the time-dependent solution of the model converges strongly to the static (steady-state) solution of the model. In this paper, we mainly consider structure of the time-dependent solution of the model. The structure of the time-dependent solution of the model were decided by spectral construction of the operator corresponding to the model, that is, distribution of spectrum of the operator. By applying the idea in [3] and the result in [2], we will prove that the C_0-semigroup generated by the operator is a quasi-compact operator. Thus we will give main spectral structure of the operator, then by combining the results in [5] and [7] with our results, we will obtain that the C_0-semigroup generated by the operator converges exponentially to a project operator. Last, for a special case, we will deduce the main results obtained in [7].

For simplicity, we introduce a notation as follows

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_{N-k} & 0 & 0 & \cdots & 0 \\ d_{01} & d_{11} & \cdots & d_{N-k-1,1} & d_{N-k,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ d_{0M} & d_{1M} & \cdots & d_{N-k-1,M} & d_{N-k,M} & 0 & \cdots & 0 \end{pmatrix}.$$

Take state space X as follows

$$X = \left\{ p \in \overbrace{R \times R \times \cdots \times R}^{N-k+1} \times \underbrace{L^1[0,\infty) \times L^1[0,\infty) \times \cdots \times L^1[0,\infty)}_{M+1} \,\middle|\, \|p\| = \sum_{i=0}^{N-k} |p_i| + \sum_{j=0}^{M} \|p_{N-k+1+j}\|_{L^1[0,\infty)} \right\}.$$

It is obvious that X is a Banach space.

Define operator A and its domain as follows

$$D(A) = \left\{ p \in X \,\middle|\, \begin{array}{l} \frac{\mathrm{d}p_j(x)}{\mathrm{d}x} \in L^1[0,\infty), \quad p_j(x) \text{ is absolutely continuous} \\ (j = N-k+1, \cdots, N-k+1+M) \quad \text{and} \quad p(0) = \Gamma p(x) \end{array} \right\},$$

$$A\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-k} \\ p_{N-k+1} \\ \vdots \\ p_{N-k+1+M} \end{pmatrix}(x)$$

$$= \begin{pmatrix} -h_0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -h_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -h_{N-k} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\frac{\mathrm{d}}{\mathrm{d}x} - \mu_{N-k+1}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \cdots & \vdots & 0 & \cdots & -\frac{\mathrm{d}}{\mathrm{d}x} - \mu_{N-k+1+M}(x) \end{pmatrix}$$

$$\times \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-k} \\ p_{N-k+1}(x) \\ \vdots \\ p_{N-k+1+M}(x) \end{pmatrix}.$$

For $\forall P \in X$, define B and E as follows

$$B\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-k} \\ p_{N-k+1} \\ \vdots \\ p_{N-k+1+M} \end{pmatrix}(x)$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{N-k-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-k} \\ p_{N-k+1}(x) \\ \vdots \\ p_{N-k+1+M}(x) \end{pmatrix},$$

$$E \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-k} \\ p_{N-k+1} \\ \vdots \\ p_{N-k+1+M} \end{pmatrix} (x) = \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{N-k} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-k} \\ p_{N-k+1}(x) \\ \vdots \\ p_{N-k+1+M}(x) \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{M} \int_0^\infty p_{N-k+1+j}(x,t)\mu_{N-k+1+j}(x)\mathrm{d}x \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Then the system of equations (1)-(7) can be written as an abstract Cauchy problem in the Banach space X:

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = (A+B+E)p(t), \quad t \in [0,\infty), \tag{8}$$

$$p(0) = (1, 0, \cdots\cdots, 0). \tag{9}$$

In this paper, firstly we consider the expression of the solution of a simple system derived from the system (8)-(9), that is, we give the expression of the C_0-semigroup generated by the operator corresponding to the simple system. Secondly, we will prove the C_0-semigroup is a quasi-compact operator. Thirdly, by using perturbation theory we will obtain that the C_0-semigroup generated by $A+B+E$ is a quasi compact operator. Lastly, by summing up all the above results and results in [7] we will deduce that the C_0-semigroup converges exponentially to a project operator. For a special case, we deduce asymptotic property of the time-dependent solution of the system.

In [7], the authors obtained the following result.

Theorem 1. *$A+B+E$ generates a positive contraction C_0- semigroup $T(t)$ and 0 is an eigenvalue of $A+B+E$ with geometric multiplicity one.*

Through investigating the proof process of Theorem 2.1 in [7], we saw easily the following result.

Theorem 2. *A generates a positive contraction C_0-semigroup $S(t)$.*

2. Main Results

Lemma 1. *If $p(x,t) = S(t)\phi(x)$ is the solution of the following system*

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = Ap(t), \quad t \in [0,\infty) \tag{10}$$

$$p(0) = \phi(x), \tag{11}$$

then

$$p(x,t) = \begin{cases} \begin{pmatrix} \phi_0 e^{-h_0 t} \\ \phi_1 e^{-h_1 t} \\ \vdots \\ \phi_{N-k} e^{-h_{N-k} t} \\ p_{N-k+1}(0, t-x) e^{-\int_0^x \mu_{N-k+1}(\tau)\mathrm{d}\tau} \\ \vdots \\ p_{N-k+1+M}(0, t-x) e^{-\int_0^x \mu_{N-k+1+M}(\tau)\mathrm{d}\tau} \end{pmatrix} & \text{as} \quad x < t, \\ \begin{pmatrix} \phi_0 e^{-h_0 t} \\ \phi_1 e^{-h_1 t} \\ \vdots \\ \phi_{N-k} e^{-h_{N-k} t} \\ \phi_{N-k+1}(x-t) e^{-\int_{x-t}^x \mu_{N-k+1}(\tau)\mathrm{d}\tau} \\ \vdots \\ \phi_{N-k+1+M}(x-t) e^{-\int_{x-t}^x \mu_{N-k+1+M}(\tau)\mathrm{d}\tau} \end{pmatrix} & \text{as} \quad x > t. \end{cases}$$

Here $p_{N-k+1}(0, t-x), \cdots, p_{N-k+1+M}(0, t-x)$ are given by (5)-(6).

Proof. Since $p(x,t) = S(t)\phi(x)$ is the solution of the system (10)-(11), $p(x,t)$ satisfies

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = -h_i p_i(t), \quad i = 0, 1, \cdots, N-k, \tag{12}$$

$$\frac{\partial p_j(x,t)}{\partial t} + \frac{\partial p_j(x,t)}{\partial x} = -\mu_j(x) p_j(x,t),$$
$$j = N-k+1, \cdots, N-k+1+M, \tag{13}$$

$$p_{N-k+1}(0,t) = a_{N-k} p_{N-k}(t), \tag{14}$$

$$p_{N-k+1+n}(0,t) = \sum_{i=0}^{N-k} d_{i,n} p_i(t), \quad n = 1, \cdots, M, \tag{15}$$

$$p_i(0) = \phi_i, \quad i = 1, \cdots, N-k,$$
$$p_j(x,0) = \phi_j(x), \quad j = N-k+1, \cdots, N-k+1+M. \tag{16}$$

If we define $\xi = x - t$ and $Q_j(t) = p_j(\xi + t, t), \quad j = N-k+1, \cdots, N-k+1+M$, then from (13) we deduce

$$\frac{\mathrm{d}Q_j(t)}{\mathrm{d}t} = -\mu_j(\xi + t) Q_j(t), \quad j = N-k+1, \cdots, N-k+1+M. \tag{17}$$

If $\xi < 0 \quad$ (i.e., $\quad x < t$), then by integrating (17) from $-\xi$ to t and using $Q_j(-\xi) = p_j(0, -\xi) = p_j(0, t-x), j = N-k+1, \cdots, N-k+1+M$, and by using new integral variable $y = \xi + \tau$ we obtain

$$\begin{aligned} p_j(x,t) &= Q_j(t) = Q_j(-\xi) e^{-\int_{-\xi}^{t} \mu_j(\xi+\tau)\,\mathrm{d}\tau} = p_j(0, t-x) e^{-\int_0^{\xi+t} \mu_j(y)\,\mathrm{d}y} \\ &= p_j(0, t-x) e^{-\int_0^x \mu_j(y)\,\mathrm{d}y}, \quad j = N-k+1, \cdots, N-k+1+M. \end{aligned} \tag{18}$$

If $\xi > 0 \quad$ (i.e., $\quad x > t$), then by integrating (17) from 0 to t and using $Q_j(0) = p_j(\xi, 0) = \phi_j(\xi) = \phi_j(x-t)$, $j = N-k+1, \cdots, N-k+1+M$, we derive

$$\begin{aligned} p_j(x,t) &= Q_j(t) = Q_j(0) e^{-\int_0^t \mu_j(\xi+\tau)\,\mathrm{d}\tau} \\ &= \phi_j(x-t) e^{-\int_\xi^{\xi+t} \mu_j(y)\,\mathrm{d}y} = \phi_j(x-t) e^{-\int_{x-t}^{x} \mu_j(y)\,\mathrm{d}y}. \end{aligned} \tag{19}$$

By solving (12) we have

$$p_i(t) = \phi_i e^{-h_i t}, \ i = 0, 1, \cdots, N-k. \tag{20}$$

From (18), (19) and (20) we know that the result of Lemma 1 is right. □

For $\phi \in X$, we define two operators as follows:

$$(U(t)\phi)(x) = \begin{cases} 0 & x \in [0,t), \\ (S(t)\phi)(x) & x \in [t,\infty), \end{cases} \tag{21}$$

$$(V(t)\phi)(x) = \begin{cases} (S(t)\phi)(x) & x \in [0,t), \\ 0 & x \in [t,\infty). \end{cases} \tag{22}$$

In [2], the author proved the following result.

Theorem 3. *A bounded and closed subset Y of*

$$X = \Bigg\{ p \in \overbrace{R \times R \times \cdots \times R}^{N-k+1} \times \underbrace{L^1[0,\infty) \times L^1[0,\infty) \times \cdots \times L^1[0,\infty)}_{M+1} \Bigg| \\ \|p\| = \sum_{i=0}^{N-k} |p_i| + \sum_{j=0}^{M} \|p_{N-k+1+j}\|_{L^1[0,\infty)} \Bigg\}$$

is compact if and only if the following two conditions hold simultaneously.

(1) $\sum_{j=N-k+1}^{N-k+1+M} \lim_{h\to 0} \int_0^\infty |\phi_j(x+h) - \phi_j(x)|\, \mathrm{d}x = 0$, *uniformly for* $\phi = (\phi_0, \phi_1, \cdots, \phi_{N-k+1+M}) \in Y$.

(2) $\sum_{j=N-k+1}^{N-k+1+M} \lim_{h\to\infty} \int_h^\infty |\phi_j(x)|\ \mathrm{d}x = 0$, *uniformly for* $\phi = (\phi_0, \phi_1, \cdots, \phi_{N-k+1+M}) \in Y$.

Theorem 4. *$V(t)$ is a compact operator in X.*

Proof. From the definition of $V(t)$ and Theorem 3, we know that it is sufficient to prove the condition (1) in Theorem 3. By Lemma 1 we have

$$\sum_{j=N-k+1}^{N-k+1+M} \int_0^t \big|p_j(x+h,t) - p_j(x,t)\big|\, \mathrm{d}x$$

$$= \sum_{j=N-k+1}^{N-k+1+M} \int_0^t \Big|p_j(0,t-x-h)e^{-\int_0^{x+h}\mu_j(\tau)\mathrm{d}\tau} - p_j(0,t-x)e^{-\int_0^{x}\mu_j(\tau)\mathrm{d}\tau}\Big|\, \mathrm{d}x$$

$$= \sum_{j=N-k+1}^{N-k+1+M} \int_0^t \Big|p_j(0,t-x-h)e^{-\int_0^{x+h}\mu_j(\tau)\mathrm{d}\tau} - p_j(0,t-x-h)e^{-\int_0^{x}\mu_j(\tau)\,\mathrm{d}\tau}$$

$$+ p_j(0,t-x-h)e^{-\int_0^x \mu_j(\tau)\,\mathrm{d}\tau} - p_j(0,t-x)e^{-\int_0^x \mu_j(\tau)\,\mathrm{d}\tau}\Big|\,\mathrm{d}x$$
$$\leq \sum_{j=N-k+1}^{N-k+1+M}\int_0^t \big|p_j(0,t-x-h)\big|\,\Big|e^{-\int_0^{x+h}\mu_j(\tau)\mathrm{d}\tau} - e^{-\int_0^x \mu_j(\tau)\mathrm{d}\tau}\Big|\,\mathrm{d}x$$
$$+\sum_{j=N-k+1}^{N-k+1+M}\int_0^t \big|p_j(0,t-x-h)-p_j(0,t-x)\big|e^{-\int_0^x \mu_j(\tau)\mathrm{d}\tau}\,\mathrm{d}x. \quad (23)$$

In the following, we will estimate first term in (23). By noting (14), (15) and using Lemma 1, we have, for $t-x-h \geq 0$

$$\big|p_{N-k+1}(0,t-x-h)\big| = \big|a_{N-k}p_{N-k}(t-x-h)\big| = \big|a_{N-k}\,\phi_{N-k}e^{-h_{N-k}(t-x-h)}\big|$$
$$\leq a_{N-k}\big|\phi_{N-k}\big|e^{-h_{N-k}(t-x-h)} \leq a_{N-k}\big|\phi_{N-k}\big| \leq \big|\phi_{N-k}\big|, \quad (24)$$

$$\big|p_{N-k+1+j}(0,t-x-h)\big| = \left|\sum_{i=0}^{N-k} d_{i,j}p_i(t-x-h)\right| \leq \sum_{i=0}^{N-k} d_{i,j}|p_i(t-x-h)|$$
$$= \sum_{i=0}^{N-k} d_{i,j}\left|\phi_i e^{-h_i(t-x-h)}\right| \leq \sum_{i=0}^{N-k} d_{i,j}|\phi_i| \leq \sum_{i=0}^{N-k}|\phi_i|, \quad j=1,\cdots,M. \quad (25)$$

By combining (24) with (25) we derive

$$\sum_{j=N-k+1}^{N-k+1+M}\int_0^t \big|p_j(0,t-x-h)\big|\,\Big|e^{-\int_0^{x+h}\mu_j(\tau)\,\mathrm{d}\tau} - e^{-\int_0^x \mu_j(\tau)\,\mathrm{d}\tau}\Big|\,\mathrm{d}x$$
$$\leq \sum_{i=0}^{N-k}\big|\phi_i\big|\sum_{j=N-k+1}^{N-k+1+M}\int_0^t \Big|e^{-\int_0^{x+h}\mu_j(\tau)\,\mathrm{d}\tau} - e^{-\int_0^x \mu_j(\tau)\,\mathrm{d}\tau}\Big|\,\mathrm{d}x$$
$$\leq \|\phi\|_X \sum_{j=N-k+1}^{N-k+1+M}\int_0^t \Big|e^{-\int_0^{x+h}\mu_j(\tau)\,\mathrm{d}\tau} - e^{-\int_0^x \mu_j(\tau)\,\mathrm{d}\tau}\Big|\,\mathrm{d}x \longrightarrow 0,$$
$$\text{as} \quad h\longrightarrow 0, \quad \text{uniformly for} \quad \phi. \quad (26)$$

In the following we will estimate the second term in (23). By using (14), (15) and Lemma 1, we calculate

$$\big|p_{N-k+1}(0,t-x-h) - p_{N-k+1}(0,t-x)\big|$$
$$= \big|a_{N-k}\big(p_{N-k}(t-x-h)+p_{N-k}(t-x)\big)\big|$$

$$= \Big| a_{N-k}\phi_{N-k}\big(e^{-h_{N-k}(t-x-h)} - e^{-h_{N-k}(t-x)}\big)\Big|$$
$$\leq a_{N-k}\big|\phi_{N-k}\big| \, \Big|e^{-h_{N-k}(t-x-h)} - e^{-h_{N-k}(t-x)}\Big| \longrightarrow 0,$$
$$\text{as} \quad h \longrightarrow 0, \quad \text{uniformly for} \quad \phi. \quad (27)$$

$$\big|p_{N-k+1+j}(0,t-x-h) - p_{N-k+1+j}(0,t-x)\big|$$
$$= \Big| \sum_{i=0}^{N-k} d_{i,j}\big(p_i(t-x-h) - p_i(t-x)\big)\Big|$$
$$\leq \sum_{i=0}^{N-k} d_{i,j}\big|p_i(t-x-h) - p_i(t-x)\big| \leq \sum_{i=0}^{N-k} \big|p_i(t-x-h) - p_i(t-x)\big|$$
$$= \sum_{i=0}^{N-k} \big|\phi_i\big|\Big|e^{-h_i(t-x-h)} - e^{-h_i(t-x)}\Big| \longrightarrow 0,$$
$$\text{as} \quad h \longrightarrow 0, \quad \text{uniformly for} \quad \phi,\ j = 1, \cdots, M. \quad (28)$$

From (27) and (28) we derive

$$\sum_{j=N-k+1}^{N-k+1+M} \int_0^t \mid p_j(0,t-x-h) - p_j(0,t-x) \mid e^{-\int_0^x \mu_j(\tau)\,\mathrm{d}\tau}\,\mathrm{d}x \longrightarrow 0,$$
$$\text{as} \quad h \longrightarrow 0, \quad \text{uniformly for} \quad \phi. \quad (29)$$

When $x \in (0,t)$, $x+h \in (0,t)$, by (26), (29) and (23), we know that

$$\sum_{j=N-k+1}^{N-k+1+M} \int_0^t \mid p_j(x+h,t) - p_j(x,t) \mid \mathrm{d}x \longrightarrow 0,$$
$$\text{as} \quad h \longrightarrow 0, \quad \text{uniformly for} \quad \phi. \quad (30)$$

If $h \in (-t,0)$, $x \in [0,t)$, then from $p_j(x+h,0) = 0$ for $x+h < 0$, $j = N-k+1, \cdots, N-k+1+M$, we deduce

$$\sum_{j=N-k+1}^{N-k+1+M} \int_0^t \mid p_j(x+h,t) - p_j(x,t) \mid \mathrm{d}x$$
$$= \sum_{j=N-k+1}^{N-k+1+M} \int_{-h}^t \big|p_j(x+h,t) - p_j(x,t) \mid \mathrm{d}x$$

$$+\sum_{j=N-k+1}^{N-k+1+M}\int_0^{-h}\big|p_j(x+h,t)-p_j(x,t)\,\big|\,\mathrm{d}x. \qquad (31)$$

Since $x+h\in[0,t)$ for $x\in[-h,t)\,,h\in[-t,0)$, by (30) it follows that

$$\sum_{j=N-k+1}^{N-k+1+M}\int_{-h}^{t}\big|p_j(x+h,t)-p_j(x,t)\,\big|\,\mathrm{d}x\longrightarrow 0,$$
$$\text{as}\quad |h|\longrightarrow 0,\quad \text{uniformly for}\quad \phi. \qquad (32)$$

By using Lemma 1, from (24) and (25) we estimate

$$\int_0^{-h}\big|p_{N-k+1}(x,t)\,\big|\,\mathrm{d}x=\int_0^{-h}\big|\,p_{N-k+1}(0,t-x)\,\big|\,e^{-\int_0^x\mu_{N-k+1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}x$$
$$\le a_{N-k}\big|\phi_{N-k}\big|\int_0^{-h}e^{-\int_0^x\mu_{N-k+1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}x\longrightarrow 0,$$
$$\text{as}\quad |h|\longrightarrow 0,\quad \text{uniformly for}\quad \phi. \qquad (33)$$

For $j=N-k+2,\cdots,N-k+1+M$, we have

$$\int_0^{-h}\big|p_j(x,t)\big|\,\mathrm{d}x=\int_0^{-h}\big|p_j(0,t-x)\big|e^{-\int_0^x\mu_j(\tau)\,\mathrm{d}\tau}\,\mathrm{d}x$$
$$\le\sum_{i=0}^{N-k}d_{ij}\big|\phi_i\big|\int_0^{-h}e^{-\int_0^x\mu_j(\tau)\,\mathrm{d}\tau}\,\mathrm{d}x\longrightarrow 0,$$
$$\text{as}\quad |h|\longrightarrow 0,\quad \text{uniformly for}\quad \phi. \qquad (34)$$

By summing up (31) – (34), it follows that, for $x\in(0,t)\,,h\in(-t,0)$,

$$\sum_{j=N-k+1}^{N-k+1+M}\int_0^{t}\big|p_j(x+h,t)-p_j(x,t)\big|\,\mathrm{d}x\longrightarrow 0,$$
$$\text{as}\quad |h|\longrightarrow 0,\quad \text{uniformly for}\quad \phi. \qquad (35)$$

From (30) and (35) we know that the result of Theorem 3 is right. □

Theorem 5. *Assume that there exist two positive constants* $\overline{\mu}\,,\underline{\mu}$ *such that* $0<\underline{\mu}\le\mu(x)\le\overline{\mu}<\infty$ *, then*

$$\|U(t)\phi\|_X\le e^{-\min\{\underline{\mu}\,,h_0\,,h_1\,,\cdots\,,h_{N-k}\}t}\|\phi\|_X.$$

Proof. For any $\phi \in X$, from the definition of $U(t)$ and Lemma 1, we estimate

$$\begin{aligned}
\|U(t)\phi\|_X &= \sum_{i=0}^{N-k} |p_i(t)| + \sum_{j=N-k+1}^{N-k+1+M} \|p_j(\cdot,t)\|_{L^1[0,\infty)} \\
&= \sum_{i=0}^{N-k} |p_i(t)| + \sum_{j=N-k+1}^{N-k+1+M} \int_t^\infty |p_j(x,t)| \, \mathrm{d}x \\
&\le \sum_{i=0}^{N-k} |\phi_i| e^{-h_i t} + \sum_{j=N-k+1}^{N-k+1+M} \int_t^\infty |\phi_j(x-t)| e^{-\int_{x-t}^x \mu_j(\tau)\,\mathrm{d}\tau} \, \mathrm{d}x \\
&\le \sum_{i=0}^{N-k} |\phi_i| e^{-\min\{h_0,\cdots,h_{N-k}\}t} + \sum_{j=N-k+1}^{N-k+1+M} \int_t^\infty |\phi_j(x-t)| e^{-\underline{\mu}t} \, \mathrm{d}x \\
&= e^{-\min\{h_0,\cdots,h_{N-k}\}t} \sum_{i=0}^{N-k} |\phi_i| + \sum_{j=N-k+1}^{N-k+1+M} \|\phi_j\|_{L^1[0,\infty)} e^{-\underline{\mu}t} \\
&\le e^{-\min\{\underline{\mu},h_0,\cdots,h_{N-k}\}t} \left\{ \sum_{i=0}^{N-k} |\phi_i| + \sum_{j=N-k+1}^{N-k+1+M} \|\phi_j\|_{L^1[0,\infty)} \right\} \\
&= e^{-\min\{\underline{\mu},h_0,\cdots,h_{N-k}\}t} \|\Phi\|_X.
\end{aligned}$$

Which shows that Theorem 4 holds. □

From Theorem 4 we derive

$$\|S(t) - V(t)\| = \|U(t)\| \le e^{-\min\{\underline{\mu},h_0,\cdots,h_{N-k}\}t}.$$

From this together with Definition 2.7 in [5, p. 214], we deduce the following result.

Theorem 6. *Assume that $\mu_j(x)$ satisfy $0 < \underline{\mu} \le \mu_j(x) \le \overline{\mu} < \infty$ for $j = N-k+1, \cdots, N-k+1+M$, then $S(t)$ is a quasi-compact operator in X.*

It is easy to see that $B : R^{N-K-1} \to R^{N-K-1}$ and $E : X \to R^{N-K-1}$ are bounded linear operators, therefore B and E are compact operators in X. From Theorem 5 and Proposition 2.9 in [5, p. 215], we conclude the following result.

Corollary 1. *Assume that $\mu_j(x)$ satisfy $0 < \underline{\mu} \le \mu_j(x) \le \overline{\mu} < \infty$ for $j = N-k+1, \cdots, N-k+1+M$, then $T(t)$ is a quasi-compact operator in X.*

From [3] we know that X^*, dual space of X, is as follows

$$X^* = \left\{ q^* \in \overbrace{R \times R \times \cdots \times R}^{N-k+1} \times \underbrace{L^\infty[0,\infty) \times \cdots \times L^\infty[0,\infty)}_{M+1} \right|$$

$$\left. |||q^*||| = \sup\Big\{ \sup_{0 \le i \le N-k} |q_i^*|, \ \sup_{0 \le j \le M} \|q^*_{N-k+1+j}(x)\|_{L^\infty[0,\infty)} \Big\} \right\}$$

$$D\Big((A+B+E)^*\Big)$$
$$= \Big\{ \frac{\mathrm{d}q_j^*(x)}{\mathrm{d}x} \text{ exists and } \ q_j^*(\infty) = \alpha, \ j = N-k+1, \cdots, N-k+1+M \Big\}.$$

Here α in $D\Big((A+B+E)^*\Big)$ has no relation with j.

Lemma 2. *$(A+B+E)^*$, the adjoint operator of $A+B+E$, is as follows*

$$(A+B+E)^* q^* = (\mathcal{L} + \mathcal{N} + \mathcal{R})\, q^*, \tag{36}$$

where $\mathcal{L}q^*(x) = Hq^*(x), \quad \mathcal{N}q^*(x) = Wq^*(x), \quad \mathcal{R}q^*(x) = Fq^*(0).$

$$H = \begin{pmatrix}
-h_0 & a_0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
b_1 & -h_1 & a_1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & b_{N-k-1} & -h_{N-k-1} & a_{N-k-1} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & b_{N-k} & -h_{N-k} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}x} - \mu_{N-k+1}(x) & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}x} - \mu_{N-k+2}(x) & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{\mathrm{d}}{\mathrm{d}x} - \mu_{N-k+1+M}(x)
\end{pmatrix},$$

$$W=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \mu_{N-k+1}(x) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mu_{N-k+1+M}(x) & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$F=\begin{pmatrix} 0 & \cdots & 0 & 0 & d_{01} & d_{02} & \cdots & d_{0M} \\ 0 & \cdots & 0 & 0 & d_{11} & d_{12} & \cdots & d_{1M} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{N-k} & d_{N-k,1} & d_{N-k,2} & \cdots & d_{N-k,M} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Proof. For any $p \in D(A)$ and $q^* = \Big(q_0^*, \cdots, q_{N-k}^*, q_{N-k+1}^*, \cdots, q_{N-k+1+M}^*\Big) \in D\big((A+B+E)^*\big)$ we have, by using integration by parts,

$$\begin{aligned} \langle (A+B+E)p, q^*\rangle &= q_0^*\Big(-h_0p_0 + b_1p_1 + \sum_{j=N-k+1}^{N-k+1+M} \int_0^\infty p_j(x)\mu_j(x)\mathrm{d}x\Big) \\ &\quad + \sum_{i=1}^{N-k-1} q_i^*\Big(a_{i-1}p_{i-1} - h_ip_i + b_{i+1}p_{i+1}\Big) \\ &\quad + q_{N-k}^*\Big(a_{N-k-1}p_{N-k-1} - h_{N-k}p_{N-k}\Big) \\ &\quad + \sum_{j=N-k+1}^{N-k+1+M} \int_0^\infty \Big(-\frac{\mathrm{d}p_j(x)}{\mathrm{d}x} - \mu_j(x)p_j(x)\Big) q_j^*(x)\mathrm{d}x \\ &= p_0\Big(-h_0q_0^* + a_0q_1^*\Big) + \sum_{i=1}^{N-k-1} p_i\Big(b_iq_{i-1}^* - h_iq_i^* + a_iq_{i+1}^*\Big) \\ &\quad + p_{N-k}\Big(b_{N-k}q_{N-k-1}^* - h_{N-k}q_{N-k}^*\Big) + q_0^* \sum_{j=N-k+1}^{N-k+1+M} \int_0^\infty p_j(x)\mu_j(x)\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty \frac{\mathrm{d}p_j(x)}{\mathrm{d}x}q_j^*(x)\mathrm{d}x-\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty \mu_j(x)p_j(x)q_j^*(x)\mathrm{d}x\\
&=p_0\Big(-h_0q_0^*+a_0q_1^*\Big)+\sum_{i=1}^{N-k-1}p_i\Big(b_iq_{i-1}^*-h_iq_i^*+a_iq_{i+1}^*\Big)\\
&+p_{N-k}\Big(b_{N-k}q_{N-k-1}^*-h_{N-k}q_{N-k}^*\Big)-\sum_{j=N-k+1}^{N-k+1+M}\Big[p_j(x)q_j^*(x)\Big]\Big|_0^\infty\\
&+\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty p_j(x)\Big[\frac{\mathrm{d}q_j^*(x)}{\mathrm{d}x}-\mu_j(x)q_j^*(x)\Big]\mathrm{d}x+\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty p_j(x)\mu_j(x)q_0^*\mathrm{d}x\\
&=p_0\Big(-h_0q_0^*+a_0q_1^*\Big)+\sum_{i=1}^{N-k-1}p_i\Big(b_iq_{i-1}^*-h_iq_i^*+a_iq_{i+1}^*\Big)\\
&+p_{N-k}\Big(b_{N-k}q_{N-k-1}^*-h_{N-k}q_{N-k}^*\Big)+\sum_{j=N-k+1}^{N-k+1+M}p_j(0)q_j^*(0)\\
&+\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty p_j(x)\Big[\frac{\mathrm{d}q_j^*(x)}{\mathrm{d}x}-\mu_j(x)q_j^*(x)\Big]\mathrm{d}x+\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty p_j(x)\mu_j(x)q_0^*\mathrm{d}x\\
&=p_0\Big(-h_0q_0^*+a_0q_1^*\Big)+\sum_{i=1}^{N-k-1}p_i\Big(b_iq_{i-1}^*-h_iq_i^*+a_iq_{i+1}^*\Big)\\
&+p_{N-k}\Big(b_{N-k}q_{N-k-1}^*-h_{N-k}q_{N-k}^*\Big)+q_{N-k+1}^*(0)a_{N-k}p_{N-k}\\
&+\sum_{j=1}^{M}q_{N-k+1+j}^*(0)\Big[\sum_{i=0}^{N-k}d_{ij}p_i\Big]\\
&+\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty p_j(x)\Big[\frac{\mathrm{d}q_j^*(x)}{\mathrm{d}x}-\mu_j(x)q_j^*(x)\Big]\mathrm{d}x+\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty p_j(x)\mu_j(x)q_0^*\mathrm{d}x\\
&=p_0\Big(-h_0q_0^*+a_0q_1^*\Big)+\sum_{i=1}^{N-k-1}p_i\Big(b_iq_{i-1}^*-h_iq_i^*+a_iq_{i+1}^*\Big)\\
&+p_{N-k}\Big(b_{N-k}q_{N-k-1}^*-h_{N-k}q_{N-k}^*\Big)\\
&+\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty p_j(x)\Big[\frac{\mathrm{d}q_j^*(x)}{\mathrm{d}x}-\mu_j(x)q_j^*(x)\Big]\mathrm{d}x+\sum_{j=N-k+1}^{N-k+1+M}\int_0^\infty p_j(x)\mu_j(x)q_0^*\mathrm{d}x
\end{aligned}
$$

$$+ p_0\Big[\sum_{j=1}^{M} d_{0j} q^*_{N-k+1+j}(0)\Big] + \sum_{i=1}^{N-k-1} p_i\Big[\sum_{j=1}^{M} d_{ij} q^*_{N-k+1+j}(0)\Big]$$
$$+ p_{N-k}\Big[a_{N-k} q^*_{N-k+1}(0) + \sum_{j=1}^{M} d_{N-k,j} q^*_{N-k+1+j}(0)\Big]$$
$$= \langle\, p\,, (\mathcal{L} + \mathcal{N} + \mathcal{R})\, q^* \,\rangle$$

Which shows that Lemma 2 is right. □

Lemma 3. *0 is an eigenvalue of $(A + B + E)^*$ with geometric multiplicity one.*

Proof. Consider $(A + B + E)^* q^* = 0$. It is equivalent to

$$-h_0 q^*_0 + a_0 q^*_1 + \sum_{j=1}^{M} d_{0j} q^*_{N-k+1+j}(0) = 0, \tag{37}$$

$$b_i q^*_{i-1} - h_i q^*_i + a_i q^*_{i+1} + \sum_{j=1}^{M} d_{ij} q^*_{N-k+1+j}(0) = 0, \quad i = 1, \cdots, N-k-1, \tag{38}$$

$$b_{N-k} q^*_{N-k-1} - h_{N-k} q^*_{N-k} + a_{N-k} q^*_{N-k+1}(0) + \sum_{j=1}^{M} d_{N-k,j} q^*_{N-k+1+j}(0) = 0, \tag{39}$$

$$\frac{\mathrm{d}q^*_j(x)}{\mathrm{d}x} = \mu_j(x) q^*_j(x) - \mu_j(x) q^*_0, \quad j = N-k+1, \cdots, N-k+1+M, \tag{40}$$

$$q^*_j(\infty) = \alpha, \quad j = N-k+1, \cdots, N-k+1+M. \tag{41}$$

By solving (40) we have

$$q^*_j(x) = b_j e^{\int_0^x \mu_j(\tau)\mathrm{d}\tau} - e^{\int_0^x \mu_j(\tau)\mathrm{d}\tau} \int_0^x q^*_0 \mu_j(\xi) e^{-\int_0^\xi \mu_j(\tau)\mathrm{d}\tau} \mathrm{d}\xi$$
$$= b_j e^{\int_0^x \mu_j(\tau)\mathrm{d}\tau} - q^*_0 e^{\int_0^x \mu_j(\tau)\mathrm{d}\tau} \int_0^x -\mathrm{d}e^{-\int_0^\xi \mu_j(\tau)\mathrm{d}\tau}$$
$$= b_j e^{\int_0^x \mu_j(\tau)\mathrm{d}\tau} - q^*_0 e^{\int_0^x \mu_j(\tau)\mathrm{d}\tau} \Big[1 - e^{-\int_0^x \mu_j(\tau)\mathrm{d}\tau}\Big] = b_j e^{\int_0^x \mu_j(\tau)\mathrm{d}\tau}$$
$$- q^*_0 e^{\int_0^x \mu_j(\tau)\mathrm{d}\tau} + q^*_0, \quad j = N-k+1, \cdots, N-k+1+M. \tag{42}$$

By combining (41) with (42) we deduce

$$b_j = q^*_0, \quad j = N-k+1, \cdots, N-k+1+M. \tag{43}$$

By substituting (43) into (42) we obtain

$$q_j^*(x) = q_0^*, \quad j = N-k+1, \cdots, N-k+1+M. \tag{44}$$

Through substituting (44) into (37) and using $h_0 = a_0 + \sum_{j=1}^{M} d_{0j}$ we calculate

$$-h_0 q_0^* + a_0 q_1^* + \sum_{j=1}^{M} d_{0j} q_{N-k+1+j}^*(0) = 0 \Longrightarrow a_0 q_1^* = a_0 q_0^* \Longrightarrow q_1^* = q_0^*. \tag{45}$$

From (44), (45), (38) and $h_1 = a_1 + b_1 + \sum_{j=1}^{M} d_{1j}$, we have

$$b_1 q_0^* - h_1 q_1^* + a_1 q_2^* + \sum_{j=1}^{M} d_{1j} q_{N-k+1+j}^*(0) = 0$$

$$\Longrightarrow a_1 q_2^* = h_1 q_1^* - b_1 q_0^* - \sum_{j=1}^{M} d_{1j} q_0^*(0) \Longrightarrow q_2^* = q_0^*. \tag{46}$$

Similar way to (45) and (46), by using $h_n = a_n + b_n + \sum_{j=1}^{m} d_{nj}$, $n = 1, 2, \cdots, M$ and (38), (39), (45), (46) we deduce

$$q_i^* = q_0^*, \quad i = 1, 2, \cdots, N-K. \tag{47}$$

By combining (47) with (44) it follows that

$$|||q^*||| = \max\left\{ \max_{0 \le i \le N-k} |q_i^*|, \max_{0 \le j \le M} \|q_{N-k+1+j}^*\|_{L^\infty[0,\infty)} \right\} = |q_0^*| < \infty. \tag{48}$$

(48) shows that 0 is an eigenvalue of $(A+B+E)^*$. Moreover, from (44) and (46) we know that the geometric multiplicity of 0 is one. The proof of Lemma 3 is completed. □

From Theorem 1, Lemma 3, and Lemma 27 in [4] we know that the algebraic multiplicity of 0 is one.

By Theorem 1 we know that the spectral bound of $A+B+E$ is zero, that is, $s(A+B+E) = 0$. Therefore, from Lemma 3, Theorem 1, Corollary 1, and Theorem 2.1 in [5, p. 343] we conclude the following result.

Theorem 7. *If $\mu_j(x)$ satisfy $0 < \underline{\mu} \le \mu_j(x) \le \overline{\mu} < \infty$ for $j = N-k+1, \cdots, N-k+1+M$, then there exists a positive projection operator P with rank one and suitable positive constants $\delta > 0$, $M \ge 0$ such that*

$$\| T(t) - P \| \le M e^{-\delta t}.$$

Here $P = \frac{1}{2\pi i} \int_{\overline{\Gamma}} (zI - A - B - E)^{-1} \mathrm{d}z$, $\overline{\Gamma}$ is a circle with center 0 and sufficiently small radius.

By Lemma 3, Theorem 1, Corollary 1, and Theorem 2.10 in [5, p. 217], Definition 2.5 in [5, p. 172] and Theorem 2.10 in [5, p. 302] we deduce

$$\left\{\gamma \in \sigma(A+B+E) \middle| Re\gamma = 0\right\} = \left\{0\right\}$$

From this together with Theorem 14 in [4] we derive the following result that was obtained in [7].

Theorem 8. *If $\mu_j(x)$ satisfy $0 < \underline{\mu} \leq \mu_j(x) \leq \overline{\mu} < \infty$ for $j = N-k+1, \cdots, N-k+1+M$, then the time-dependent solution of the system (8)-(9) converges strongly to the steady-state solution of the system (8)-(9) as time tends to infinite, that is*

$$\lim_{t\to\infty} p(x,t) = \alpha p(x)\,.$$

Here $p(x)$ is an eigenvector corresponding to zero.

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