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OPERATORS WITH SLOWLY GROWING RESOLVENTS TOWARDS THE SPECTRUM

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Abstract: A closed densely defined operator H, on a Banach space $\mathcal{X}$, whose spectrum is contained in $\mathbb{R}$ and satisfies

$$\left\|(z-H)^{-1}\right\| \leq c\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\beta}} \qquad \forall\, z \notin \mathbb{R} \tag{0.1}$$

for some α , $\beta \geq 0$; $c > 0$, is said to be of $(\alpha,\ \beta) - type\ \mathbb{R}$. If instead of (0.1) we have

$$\left\|(z-H)^{-1}\right\| \leq c\frac{|z|^{\alpha}}{|\Im z|^{\beta}} \qquad \forall\, z \notin \mathbb{R}, \tag{0.2}$$

then H is of $(\alpha,\ \beta)' - type\ \mathbb{R}$.

Examples of such operators include self-adjoint operators, Laplacian on $L^1(\mathbb{R})$, Schrödinger operators on $L^p(\mathbb{R}^n)$ and operators H whose spectra lie in $\mathbb{R}$ and permit some control on $\left\|e^{iHt}\right\|$.

In this paper we will characterise the $(\alpha,\ \beta) - type\ \mathbb{R}$ operators. In particular we show that property (0.1) is stable under dialation by real numbers in the interval (0,1) and perturbation by positive reals. We will also show that is H is of $(\alpha,\ \beta) - type\ \mathbb{R}$ then so is H^2.

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1. Preliminaries

Suppose H is a closed densely defined operator on a Banach space $\mathcal{X}$ whose spectrum is contained in $\mathbb{R}$ and there exists $c > 0$ such that

$$\left\|(z-H)^{-1}\right\| \;\leq\; c\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\beta}} \tag{1.1}$$

for all $z \notin \mathbb{R}$ and some $\alpha,\ \beta \geq 0$. We will say that H is of $(\alpha,\ \beta) - type\ \mathbb{R}$. Here, we define $\langle . \rangle$ by $\langle z\rangle^2 := 1 + |z|^2$ and $\Im z$ denotes the *imaginary part of* z (the *real part of* z will be denoted by $\Re z$). If instead of (1.1) we have

$$\left\|(z-H)^{-1}\right\| \;\leq\; c\frac{|z|^{\alpha}}{|\Im z|^{\beta}} \tag{1.2}$$

for all $z \notin \mathbb{R}$ and some $\alpha,\ \beta \geq 0$, we will say that H is of $(\alpha,\ \beta)' - type\ \mathbb{R}$.

1.1. Examples of $(\alpha,\ \beta) - type\ \mathbb{R}$ Operators

Proposition 1.1. *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then H is of $(0,\ 1) - type\ \mathbb{R}$.*

Proof. If $S \subset \mathcal{H}$, denote the closure of S by $\overline{S}$ and its orthogonal complement by $S^{\perp}$. Let the adjoint of an operator H be denoted by H^*, its *kernel* be $\ker(H) := \{f \in \mathfrak{D}(H)$ such that $Hf = 0\}$ and suppose $z \in \mathbb{C} \setminus \mathbb{R}$, then

$$\begin{aligned} 2i\Im\langle (z-H)f, f\rangle &= \langle (z-H)f, f\rangle - \overline{\langle (z-H)f, f\rangle} \\ &= 2i\Im z\,\|f\|^2 \\ \text{if and only if } \Im\langle f, (z-H)f\rangle &= \Im z\,\|f\|^2 . \\ \text{Which implies } |\Im z|\,\|f\| &\leq \|(z-H)f\| . \end{aligned} \tag{1.3}$$

That is, $z - H$ is bounded from below and hence $z - H$ is injective, and so $\ker(z - H) = \{0\}$. Since H is self adjoint, we have

$$\ker((z-H)^*) = \ker(\bar{z} - H) = \{0\}. \tag{1.4}$$

But because H is closed densely defined,

$$\overline{\Re(z-H)} = \ker((z-H)^*)^{\perp} .$$

Therefore using (1.4),

$$\overline{\Re(z-H)} \quad = \quad \{0\}^{\perp} \quad = \quad \mathcal{H}.$$

Conclusion:

1. $(z-H)^{-1}$ exists and is bounded,
2. $\Re(z-H)$ is dense in $\mathcal{H}$

thus $z \in \rho(H)$.

The conclusion of the proposition now follows from (1.3). $\square$

Proposition 1.2. *Let* $H_0 = -\frac{d^2}{dx^2}$ *on* $L^1(\mathbb{R})$ *where* $\mathfrak{D}(H_0) = \{f \in L^1(\mathbb{R}) \ : \ f'' \in L^1(\mathbb{R}), f'$ *absolutely continuous*$\}$. *Then*

1. $(z - H_0)^{-1}$ *is a convolution operator, for each* $z \notin \mathbb{R}$.
2. H_0 *is of* $(0,\ 1) - type\ \mathbb{R}$.

Proof. Let $z \in \rho(H_0)$. Then

$$\widehat{(z - H_0)^{-1} f}(\zeta) = (z - a(\zeta))^{-1} \widehat{f}(\zeta),\ \zeta \in \mathbb{R}$$

where $\widehat{g}$ denotes Fourier transform of g and the symbol of H_0, $a(\zeta) = \zeta^2$. So that $\widehat{(z - H_0)^{-1} f}(\zeta) = \widehat{g * f}(\zeta)$ where $g * f$ denotes the convolution of g and f. Observe that

$$\widehat{g}(\zeta) := \frac{(2\pi)^{-1/2}}{z - \zeta^2}$$

decays rapidly enough as $|\zeta| \ \to \ \infty, \quad \widehat{g} \in L^1(\mathbb{R})$.

$$\text{Thus} \quad (z - H_0)^{-1} f(x) = g * f(x), \qquad f \in L^1(\mathbb{R})$$

where $g \in C_0(\mathbb{R})$. In fact $g(x) = \frac{i}{2\sqrt{z}} e^{-i\sqrt{z}|x|}$ explicitly.

This proves the first part of the proposition.

Next

$$\begin{aligned} \left\|(z - H_0)^{-1}\right\| &= \sup\left\{\|g * f\|_1 \ : \ f \in L^1, \|f\|_1 = 1\right\} \\ &\leq \|g\|_1 \\ &\quad (\text{since } \|g * f\|_1 \leq \|g\|_1 \|f\|_1 \, f, g \in L^1) \\ &= \int_{-\infty}^{\infty} \left| \frac{i}{2\sqrt{z}} e^{-i\sqrt{z}|x|} \right| dx. \end{aligned}$$

By means of a change of variable and reflectional symmetry we conclude that

$$\left\|(z - H_0)^{-1}\right\| = 2 \int_0^{\infty} \left| \frac{i}{2\sqrt{z}} e^{-i\sqrt{z}|x|} \right| dx \leq \frac{2}{|\Im z|}.$$ $\square$

2. Main Results

Theorem 2.1. *Let* H *be a bounded operator with* $\sigma(H) \subseteq \mathbb{R}$, *and*

$$\left\|e^{iHt}\right\| \leq C(1 + |t|)^{\alpha}, \tag{2.1}$$

where α *is an non-negative integer. Then* H *is of* $(\alpha,\ \alpha + 1) - type\ \mathbb{R}$.

Proof. Let $z \notin \mathbb{R}$. Since $(z - iH)^{-1} = \begin{cases} \int_0^\infty e^{-zt} e^{iHt} dt & \Re z > 0 \\ -\int_{-\infty}^0 e^{-zt} e^{iHt} dt & \Re z < 0 \end{cases}$ (see Bratteli and Robinson Proposition 3.16, [1]). Therefore for z with $\Re z > 0$ we have

$$\left\|(z - iH)^{-1} f\right\| \le \int_0^\infty e^{-|\Re z| t} \left\|e^{iHt} f\right\| dt \le C \|f\| \int_0^\infty e^{-|\Re z| t} (1+t)^\alpha dt.$$

If $\Re z < 0$, put $s = -t$ then

$$\left\|-(z - iH)^{-1} f\right\| \le \|f\| \int_0^\infty |e^{zs}| \left\|e^{-iHs}\right\| ds \le \int_0^\infty e^{-|\Re z| s} \left\|e^{-iHs} f\right\| ds$$
$$\le C \|f\| \int_0^\infty e^{-|\Re z| s} (1+s)^\alpha ds.$$

But

$$\int_0^\infty e^{-|\Re z| t} (1+t)^\alpha dt = \left[-\frac{1}{|\Re z|} e^{-|\Re z| t} (1+t)^\alpha\right]_0^\infty$$
$$-\int_0^\infty -\frac{1}{|\Re z|} e^{-|Rez| t} \alpha (1+t)^{\alpha-1} dt = \frac{1}{|\Re z|} + \frac{\alpha}{|\Re z|^2} + \frac{\alpha(\alpha-1)}{|\Re z|^3} + \ldots + \frac{\alpha!}{|Rez|^{\alpha+1}}.$$

Consequently we may conclude that

$$\begin{aligned} \left\|(z - iH)^{-1} f\right\| &\le C \|f\| \frac{1}{|\Re z|} \left[1 + \frac{\alpha}{|\Re z|^1} + \frac{\alpha(\alpha-1)}{|\Re z|^2} + \ldots + \frac{\alpha!}{|\Re z|^\alpha}\right] \\ &\le C \|f\| \frac{\alpha!}{|\Re z|} \left[1 + \frac{\alpha}{1! |\Re z|} + \frac{\alpha(\alpha-1)}{2! |\Re z|^2} + \ldots + \frac{\alpha!}{\alpha! |\Re z|^\alpha}\right] \\ &= C \|f\| \frac{\alpha!}{|\Re z|} (1 + \frac{1}{|\Re z|})^\alpha \le 2^{\alpha/2} C \alpha! \frac{\langle z \rangle^\alpha}{|\Re z|^{\alpha+1}} \|f\|, \end{aligned}$$

where we have used Hölder's inequality to obtain

$$1 + |\Re z| \le \sqrt{2} (1 + |\Re z|^2)^{1/2} \le \sqrt{2} \langle z \rangle .$$

Now put $w := i^{-1} z$. Then $w \notin \mathbb{R}$ and

$$\left\|(w - H)^{-1} f\right\| = \left\|i(z - iH)^{-1} f\right\| \le 2^{\alpha/2} C \alpha! \frac{\langle z \rangle^\alpha}{|\Re z|^{\alpha+1}} \|f\|$$
$$= 2^{\alpha/2} C \alpha! \frac{\langle w \rangle^\alpha}{|\Im w|^{\alpha+1}} \|f\|. \quad \square$$

Remark 2.2. 1. Note that the converse of Theorem 2.1 is false. A counter example is the following:

Let $H_0 = -\frac{d^2}{dx^2}$ on $L^1(\mathbb{R})$. Then by Proposition 1.2, $(z - H_0)^{-1}$ is a convolution operator, for each $z \notin \mathbb{R}$ and is of $(0,\ 1) - \mathit{type}\ \mathbb{R}$.

However, operators e^{iH_0t} are unbounded for all $t \neq 0$, see for example, Brenner et al [2, p. 27].

2. Since the map $z \to e^{itz}$ is in $H^\infty(\Omega_\epsilon)$, where $\Omega_\epsilon := \{z \in \mathbb{C} : |\Im z| < \epsilon\}$ for some $\epsilon > 0$, one may conjecture that the conclusion of theorem 2.1 holds even for unbounded operators whose spectra lie in Ω_ϵ and admit the bound (2.1). This problem as far as we can tell, remains open.

A Schrödinger operator operator, $H := -\Delta + V$ on $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$ considered in this study, has potential V such that $V_+ \in K_N^{loc}$ and $V_- \in K_N$, see the footnote[1]. The spectrum of the Schrödinger operator with potential chosen as outlined, is real and independent of p (see, Hempel and Voigt [5], Proposition 4.3(a)). The operators $e^{i\Delta t}$ are however unbounded on $L^p(\mathbb{R}^N)$ for all $t \neq 0,\ N \geq 1$ and for all $p \neq 2$ in the range $[1, \infty]$, Brenner et al p. 27 in [2].

Theorem 2.3. *Let H be a Schrödinger operator on $L^p(\mathbb{R}^N)$ then $H + \lambda$ is $(N,\ N+1)' - type\ \mathbb{R}$ for $\lambda > 0$ large enough.*

Proof. See Pang [8]. □

Theorem 2.4. *Let H be a Schrödinger operator on $L^p(\mathbb{R}^N)$ then H is $(\alpha,\ \alpha+1)' - type\ \mathbb{R}$ for $\alpha := N\left|\frac{1}{p} - \frac{1}{2}\right|$.*

Proof. See S. Nakamura Theorem 8, [7]. □

Theorem 2.5. *Let H be an $n \times n$ matrix with real eigenvalues. Then H is of $(n-1,\ n) - type\ \mathbb{R}$. H is of $(0,\ 1) - type\ \mathbb{R}$ if and only if it is diagonalizable.*

Proof. Consider an $n \times n$ tridiagonal matrix :

$$A = \begin{pmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & a_{n-1} & b_{n-1} \\ 0 & & & c_{n-1} & a_n \end{pmatrix} \tag{2.2}$$

with either $c_i = 0$ for all i or $b_i = 0$ for all i. Then

$$\det(A) = \prod_{i=1}^{n} a_i. \tag{2.3}$$

[1] $V_-(x) := \max\{0, V(x)\}$ and $V_+(x) := \min\{0, V(x)\}$.

Next, suppose $\lambda_1, \ldots, \lambda_n$ are the real eigenvalues of H and $z \notin \mathbb{R}$. Then H has the Jordan canonical form

$$H = \begin{pmatrix} \lambda_1 & b_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & b_{n-1} \\ 0 & & & \lambda_n \end{pmatrix},$$

where $b_\nu = 0$ or 1 for all ν. Thus

$$(z - H) = \begin{pmatrix} z-\lambda_1 & -b_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -b_{n-1} \\ 0 & & & z-\lambda_n \end{pmatrix}.$$

This is a special case of (2.2) with $c_1 = c_2 = \ldots = c_{n-1} = 0$. Hence

$$\begin{aligned} \det(z-H) &= \prod_{\nu=1}^{n} (z-\lambda_\nu) \qquad \text{(by (2.3))} \\ &\neq 0 \end{aligned}$$

(since $z \neq \lambda_\nu$ for all ν). Thus $\operatorname{cof}(z-H) = (d_{ij})$, where

$$d_{ij} = \begin{cases} (-1)^{i+j}(z-\lambda_1)\ldots(z-\lambda_{j-1})(-b_j)\ldots(-b_{i-1})(z-\lambda_{i+1})\ldots(z-\lambda_n), & i > j\,, \\ \prod_{\nu\neq i}^{n}(z-\lambda_\nu), & i = j\,, \\ 0, & i < j. \end{cases}$$

If $\|.\|_{HS}$ denotes the Hilbert-Schmidt norm and A^T is the transpose of the matrix A, then by the above together with the fact that $(z-H)^{-1} = (\det(z-H))^{-1}(\operatorname{cof}(z-H))^T$ we see that

$$\begin{aligned} \left\|(z-H)^{-1}\right\|_{HS}^2 = \left\|\left[(z-H)^{-1}\right]^T\right\|_{HS}^2 &= \left\|\frac{\operatorname{cof}(z-H)}{\det(z-H)}\right\|_{HS}^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{i}\left|\frac{\prod_{\substack{1\leq\nu\leq j-1 \\ i+1\leq\nu\leq n}}(z-\lambda_\nu)\cdot\prod_{j\leq\nu\leq i-1}(-b_\nu)}{\prod_{\nu=1}^{n}(z-\lambda_\nu)}\right|^2 \\ &\qquad \text{(since } d_{ij} = 0 \text{ for all } i<j). \end{aligned}$$

Here we have used the fact that $\|A\|_{HS}^2 = \sum_1^n \|Ae_i\|^2$ for any orthonormal system $e_1, \ldots, e_n$.

Thus

$$\left\|(z-H)^{-1}\right\|_{HS}^{2} = \sum_{i=1}^{n}\sum_{j=1}^{i}\left|\prod_{j\le\nu\le i}\frac{1}{(z-\lambda_\nu)}\cdot\prod_{j\le\nu\le i-1}(-b_\nu)\right|^2 \tag{2.4}$$

$$\le \sum_{i=1}^{n}\sum_{j=1}^{i}\prod_{j\le\nu\le i}\left|\frac{1}{z-\lambda_\nu}\right|^2 \tag{2.5}$$

(since $|-b_\nu|\le 1$ for all ν).

In this case $\sigma(H)=\{\lambda_1,\dots,\lambda_n\}$, and we can find $\lambda\in\sigma(H)$ such that

$$\operatorname{dist}(z,\ \sigma(H)) = |z-\lambda|\,. \tag{2.6}$$

So, $|z-\lambda|\le|z-\lambda_i|$ for all i and

$$\left\|(z-H)^{-1}\right\|_{HS}^{2}\le\sum_{i=1}^{n}\sum_{j=1}^{i}\left|\frac{1}{z-\lambda}\right|^{2(i-j+1)}. \tag{2.7}$$

We observe that

$$\left\|(z-H)^{-1}\right\|_{HS}\le\sqrt{\frac{n(n+1)}{2}}\cdot\frac{\langle z\rangle^{n-1}}{|\Im z|^n}\quad\text{for all } z\notin\mathbb{R}. \tag{2.8}$$

Since all norms in a finite dimensional space are equivalent, we conclude that

$$\left\|(z-H)^{-1}\right\|\le c\frac{\langle z\rangle^{n-1}}{|\Im z|^n}\quad\text{for all } z\notin\mathbb{R}\text{ and some } c>0. \tag{2.9}$$

If H is diagonalizable, then

$$(z-H)=\begin{pmatrix} z-\lambda_1 & & 0\\ & \ddots & \\ 0 & & z-\lambda_n\end{pmatrix}.$$

This is a special case of (2.2) with $c_i=b_i=0$ for all $i=2,1\dots n-1$. In this case $\operatorname{cof}(z-H)=(d_{ij})$ where

$$d_{ij}=\begin{cases} 0 & ,\quad i>j\,,\\ \prod_{\nu\ne i}^{n}(z-\lambda_\nu) & ,\quad i=j\,,\\ 0 & ,\quad i<j\,,\end{cases}$$

and

$$\left\|(z-H)^{-1}\right\|_{HS}^{2}\ \le\ \sum_{i=1}^{n}\left|\frac{\prod_{\nu\ne i}(z-\lambda_\nu)}{\prod_{\nu=1}^{n}(z-\lambda_\nu)}\right|^2\ =\ \sum_{i=1}^{n}\left|\frac{1}{z-\lambda_i}\right|^2\ \le\ \left|\frac{1}{z-\lambda}\right|^2 n$$

with λ chosen as in (2.6).

Therefore
$$\left\|(z-H)^{-1}\right\|_{HS} \leq \frac{\sqrt{n}}{|z-\lambda|} \leq \frac{\sqrt{n}}{|\Im z|}. \tag{2.10}$$
Thus,
$$\left\|(z-H)^{-1}\right\| \leq \frac{c}{|\Im z|} \text{ for some } c>0. \tag{2.11}$$

Conversely, if H is not diagonalizable, then from (2.4) we have
$$\left\|(z-H)^{-1}\right\|_{HS}^2 = \sum_{i=1}^{n}\sum_{j=1}^{i}\left|\prod_{j\leq\nu\leq i}\frac{1}{(z-\lambda_\nu)}\cdot\prod_{j\leq\nu\leq i-1}(-b_\nu)\right|^2$$
with $b_k \neq 0$ for some $k \in \{1,\dots n-1\}$. Thus
$$\left\|(z-H)^{-1}\right\|_{HS}^2 = \sum_{i=1}^{n}\left|\frac{1}{z-\lambda_i}\right|^2 + \sum_{i=2}^{n}\sum_{j=k_i}^{i-1}\left|\prod_{j\leq\nu\leq i-1}\frac{-b}{(z-\lambda_\nu)}\right|^2,$$
where $k_i := \min\{j \ : \ b_j, b_{j+1}, \dots, b_{i-1} \neq 0\}$. But $|\Im z| = |\Im(z-\lambda_i)|$ for all $i=1,\dots,n$ and $|-b_j| = 1$ for all $k_i \leq j \leq i-1$. So if we set $\Re(z+\lambda) := \max\{\Re(z-\lambda_i) \ : \ i=1,\dots,n\}$ then
$$\begin{aligned}|\Im z|^2\left\|(z-H)^{-1}\right\|_{HS}^2 &> \sum_{i=1}^{n}\frac{|\Im z|^2}{[|\Im z|+|\Re(z-\lambda)|]^2} \\ &\quad + \sum_{i=2}^{n}\frac{|\Im z|^2}{[|\Im z|+|\Re(z-\lambda)|]^2}\sum_{j=k_i}^{i-1}\frac{1}{[|\Im z|+|\Re(z+\lambda)|]^{a_j}} \\ &\quad \text{with } \ a_j \geq 2 \text{ for all } \ j. \\ &= nK + \sum_{i=2}^{n}K\sum_{j=k_i}^{i-1}\frac{1}{[|\Im z|+|\Re(z+\lambda)|]^{a_j}} \end{aligned} \tag{2.12}$$
where $K := \frac{|\Im z|^2}{[|\Im z|+|\Re(z-\lambda)|]^2}$, $K \to 0$ as $\Im z \to 0$ but $\frac{1}{[|\Im z|+|\Re(z+\lambda)|]^{a_j}} \to \infty$ a_i times faster, as $\Im z \to 0$ for any fixed $\Re(z+\lambda)$. Therefore it follows from (2.12) that there is no $D>0$ such that $|\Im z|^2\left\|(z-H)^{-1}\right\|_{HS}^2 \leq D$ for all $z \notin \mathbb{R}$. □

3. Conclusions and Applications

Theorem 3.1. *H is of $(0,\ 1)-type\ \mathbb{R}$ with the constant $c=1$ if and only if iH is a generator of a one-parameter group of isometries on $\mathcal{X}$.*

Proof. Necessity. Assume H is of $(0,\ 1) - type\ \mathbb{R}$.

Clearly $\pm iH$ are closed densely defined (by the hypothesis on H). Suppose $\lambda > 0$. Then $\lambda \in \rho(\pm iH)$ since $\sigma(H) \subset \mathbb{R}$, and

$$\left\|(\lambda \pm iH)^{-1}\right\| = \left\|\frac{1}{i}(\frac{\lambda}{i} \pm H)^{-1}\right\| = \left\|(-i\lambda \pm H)^{-1}\right\| \leq |\Im(\pm i\lambda)|^{-1} = \lambda^{-1}.$$

Thus by Hille-Yosida theorem (See Goldstein [4], p. 16-17), $\pm iH$ are generators of contraction semigroups. Finally, the conclusion follows by invoking Prop. 1.14 of Davies [3].

Sufficiency. Suppose iH is a generator of a group of isometries $\{T(t)\}$.

Then for all $w \in \mathbb{C}$ with $\Re w \neq 0$, $\quad w \in \rho(iH)$ and

$$(\lambda - iH)^{-1} = \begin{cases} \int_0^\infty T(t)e^{-\lambda t}dt & \text{if } \Re\lambda > 0 \\ -\int_\infty^0 T(t)e^{-\lambda t}dt & \text{if } \Re\lambda < 0 \end{cases}$$

(see Bratteli and Robinson, [1, Proposition 3.16]).

From this

$$\left\|(\lambda - iH)^{-1}\right\| \leq \int_0^\infty e^{-|\Re\lambda|t} \left\|T(t)\right\| dt \leq \int_0^\infty e^{-|\Re\lambda|t}dt = |\Re(\lambda)|^{-1}.$$

Now put $z := \frac{\lambda}{i}$. □

Theorem 3.2. *If H is of $(\alpha,\ \alpha+1)' - type\ \mathbb{R}$ then λH is also of $(\alpha,\ \alpha+1)' - type\ \mathbb{R}$ with the same constant c for all $\lambda > 0$.*

Proof. Let $z \notin \mathbb{R}$, then

$$\begin{aligned} \left\|(z - \lambda H)^{-1}\right\| &= \left\|\lambda^{-1}(\frac{z}{\lambda} - H)^{-1}\right\| \\ &\leq \left|\lambda^{-1}\right| c \left|\Im\frac{z}{\lambda}\right|^{-1} \left(\frac{\left|\frac{z}{\lambda}\right|}{\left|\Im\frac{z}{\lambda}\right|}\right)^\alpha \quad \text{(hypothesis)} \\ &= c\left|\Im z\right|^{-1}\left(\frac{|z|}{|\Im z|}\right)^\alpha. \end{aligned}$$

Remark 3.3. The type of stability shown in Theorem 3.2 will be called *scale invariance.*

Theorem 3.4. *If H is of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ then λH is also of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ with the same constant c for all $0 < \lambda < 1$.*

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$, then

$$\left\|(z - \lambda H)^{-1}\right\| = \left\|\lambda^{-1}(\frac{z}{\lambda} - H)^{-1}\right\| \leq \left|\lambda^{-1}\right| c \left|\Im\frac{z}{\lambda}\right|^{-1} \left(\frac{\left\langle\frac{z}{\lambda}\right\rangle}{\left|\Im\frac{z}{\lambda}\right|}\right)^\alpha \text{ (hypothesis).}$$

Thus

$$\left\|(z-\lambda H)^{-1}\right\| \leq \left|\lambda^{-1}\right| c\,|\Im z|^{-1}\,|\lambda| \left(\frac{\sqrt{\lambda^2+|z|^2}}{|\Im z|}\right)^{\alpha} \leq c\,|\Im z|^{-1}\left(\frac{\sqrt{1+|z|^2}}{|\Im z|}\right)^{\alpha}$$
$$= c\,|\Im z|^{-1}\left(\frac{\langle z\rangle}{|\Im z|}\right)^{\alpha}.$$

Remark 3.5. The type of stability shown in Theorem 3.4 will be called *scale sub-invariance*.

Theorem 3.6. *If H is of $(\alpha,\ \beta) - type\ \mathbb{R}$ then $H+\lambda$ is also of $(\alpha,\ \beta) - type\ \mathbb{R}$ for all $\lambda \in \mathbb{R}$.*

Proof. If $\lambda \in \mathbb{R}$ and $z \notin \mathbb{R}$ then $\Im(z-\lambda) = \Im z$. Therefore we get

$$\left\|(z-(H+\lambda))^{-1}\right\| = \left\|[(z-\lambda)-H]^{-1}\right\| \leq c\frac{\langle z-\lambda\rangle^{\alpha}}{|\Im(z-\lambda)|^{\beta}} \quad \text{(hypothesis)}$$
$$\leq c_1\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\beta}},$$

where $c_1 = c2^{\alpha/2}\langle\lambda\rangle^{\alpha}$. □

If H is of $(\alpha,\ \beta)' - type\ \mathbb{R}$ then it is also of $(\alpha,\ \beta) - type\ \mathbb{R}$ with the converse not true in general. However the following result provides some sort of converse to this.

Theorem 3.7. *If λH is of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ for all $\lambda > 0$, then H is of $(\alpha,\ \alpha+1)' - type\ \mathbb{R}$ with the same constant c.*

Proof. Let $z \notin \mathbb{R}$, then

$$\left\|(z-H)^{-1}\right\| = \left\|\lambda\,(\lambda z-\lambda H)^{-1}\right\|$$
$$\leq |\lambda|\,c\,|\Im(\lambda z)|^{-1}\left(\frac{\langle\lambda z\rangle}{|\Im(\lambda z)|}\right)^{\alpha} \quad \text{(hypothesis)}$$
$$= c\,|\Im z|^{-1}\left(\frac{\sqrt{\lambda^{-2}+|z|^2}}{|\Im z|}\right)^{\alpha} \quad \text{for all } \lambda > 0.$$

Letting $\lambda \to \infty$, we observe that

$$\left\|(z-H)^{-1}\right\| \leq c\,|\Im z|^{-1}\left(\frac{|z|}{|\Im z|}\right)^{\alpha}.$$ □

Conjecture 3.8. *If H is of $(\alpha,\ \beta) - type\ \mathbb{R}$, $\beta \geq \alpha$ and $0 \notin \sigma(H)$ then*

H is of $(\alpha+1,\ \beta)' - type\ \mathbb{R}$.

Theorem 3.9. *Let H be of* $(\alpha,\ \beta) - type\ \mathbb{R}$ *. Then* H^2 *is of* $(\frac{\alpha+\beta-1}{2},\beta)$ *– type* $\mathbb{R}$.

Proof. Given $z \notin \mathbb{R}$, we have

$$\frac{1}{|\Im\sqrt{z}|^2} = \frac{1}{\frac{1}{2}\,||z| - \Re z|} \le \frac{4\,|z|}{|\Im z|^2}.$$

Also since $\langle\sqrt{z}\rangle^2 \le \sqrt{2}\,\langle z\rangle$ and $2\sqrt{z}(z-H^2)^{-1} = (\sqrt{z}-H)^{-1} - (-\sqrt{z}-H)^{-1}$, we get

$$\begin{aligned}\left\|(z-H^2)^{-1}\right\| &\le \frac{1}{2}\left|\frac{1}{\sqrt{z}}\right|\left\{\left\|(\sqrt{z}-H)^{-1}\right\| + \left\|(-\sqrt{z}-H)^{-1}\right\|\right\}\\ &\le c\left|\frac{1}{\sqrt{z}}\right|\frac{\langle\sqrt{z}\rangle^\alpha}{|\Im\sqrt{z}|^\beta} \quad \text{(hypothesis)}\\ &\le \frac{c2^{\alpha/4+\beta}\,\langle z\rangle^{(\alpha+\beta-1)/2}}{|\Im z|^\beta}. \quad \square\end{aligned}$$

Proposition 3.10. *Let A be of* $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ *with*

$$\left\|(z-A)^{-1}\right\| \le c_1\frac{\langle z\rangle^\alpha}{|\Im z|^{\alpha+1}} \qquad \textit{for some } c_1 > 0 \text{ and } \alpha \ge 0$$

and B of $(\beta,\ \beta+1) - type\ \mathbb{R}$ *with*

$$\left\|(z-B)^{-1}\right\| \le c_2\frac{\langle z\rangle^\beta}{|\Im z|^{\beta+1}} \qquad \textit{for some } c_2 > 0 \text{ and } \beta \ge 0.$$

Then

$$\begin{aligned}&\left\|(z-A)^{-1} - (z-B)^{-1}\right\|\\ &\qquad \le (1+\sqrt{2}c_1)(1+\sqrt{2}c_2)\left\|(i+A)^{-1} - (i+B)^{-1}\right\|\frac{\langle z\rangle^{\alpha+\beta+2}}{|\Im z|^{\alpha+\beta+2}}.\end{aligned}$$

Proof. $(z-A)$ and $(z-A)^{-1}$ commute on $\mathfrak{D}(A)$ and hence $(i+A)$ and $(z-A)^{-1}$ also commute on $\mathfrak{D}(A)$ since by linearity of $(z-A)^{-1}$, for all $g \in \mathfrak{D}(A)$, we have

$$\begin{aligned}(z-A)^{-1}(i+A)g &= (z-A)^{-1}\left\{(z+i) - (z-A)\right\}g\\ &= (z-i)(z-A)^{-1}g - (z-A)^{-1}(z-A)g\\ &= (z-i)(z-A)^{-1}g - (z-A)(z-A)^{-1}g = (i+A)(z-A)^{-1}g.\end{aligned}$$

Also, $(i+A)^{-1}$ maps into $\mathfrak{D}(A)$ and hence $(i+A)$ and $(z-A)^{-1}$ commute on $\mathfrak{R}((i+A)^{-1})$.

The operators $D := (i+A)(z-A)^{-1}$, $H := \left[(i+A)^{-1} - (i+B)^{-1}\right]$ and $E := (i+B)(z-B)^{-1}$ are defined everywhere and bounded on $\mathcal{X}$ as can be seen by writing $D = (i+z)(z-A)^{-1} - I$ and $E = (i+z)(z-B)^{-1} - I$.

Now for $f \in \mathcal{X}$ we have

$$\begin{aligned} D&\left[(i+A)^{-1} - (i+B)^{-1}\right] Ef \\ &= (i+A)(z-A)^{-1}\left[(i+A)^{-1} - (i+B)^{-1}\right](i+B)(z-B)^{-1}f \\ &= -\left[(z-A)^{-1} - (z-B)^{-1}\right] f. \end{aligned}$$

Thus

$$\begin{aligned} \|D\| = \left\|(i+z)(z-A)^{-1} - I\right\| &\leq 1 + (1+|z|)\left\|(z-A)^{-1}\right\| \\ &\leq (1+\sqrt{2}c_1)\frac{\langle z\rangle^{\alpha+1}}{|\Im z|^{\alpha+1}}. \end{aligned}$$

Similarly, $\|E\| \leq (1+\sqrt{2}c_2)\frac{\langle z\rangle^{\beta+1}}{|\Im z|^{\beta+1}}$. Therefore

$$\left\|(z-A)^{-1} - (z-B)^{-1}\right\| \leq (1+\sqrt{2}c_1)(1+\sqrt{2}c_2)\,\|H\|\,\frac{\langle z\rangle^{\alpha+\beta+2}}{|\Im z|^{\alpha+\beta+2}}.$$

Theorem 3.11. *If H is of $(\alpha,\ \alpha+1)' - type\ \mathbb{R}$, for some $\alpha > 0$, then H admits $C_0^\infty(\mathbb{R})$ functional calculus.*

Proof. See Balabane et al [6], Theorem 4.11. □

In fact we believe that $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ and $(\alpha,\ \alpha+1)' - type\ \mathbb{R}$ operators do admit a much larger functional calculus than $C_0^\infty(\mathbb{R})$.

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