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MEROMORPHIC FUNCTIONS WITH THEIR RELATIVE DEFICIENCIES

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Abstract: In this paper we consider two meromorphic functions having common roots and find some relations involving their relative deficiency.

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1. Introduction, Definitions and Notations

Let f_1 and f_2 be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let $n_0(r,a)$ and $\bar{n_0}(r,a)$ respectively denote the number of common roots and the number of distinct common roots in the disk $|z| \le r$ of the two equations $f_1 = a$ and $f_2 = a$ where a is any complex number. Singh [2] found some relations on the relative defects corresponding to the common roots of two meromorphic functions. In the paper, we further investigate the results of Singh [2] and prove some new results on relative defects of the common roots of $f_1 = a$ and $f_2 = a$.

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To start our paper, we require the following quantities:

$$\text{Let } \bar{N}_0(r,a) = \int_0^r \frac{\overline{n}_0(t,a) - \overline{n}_0(0,a)}{t} dt + n_0(0,a) \log r$$

and $\bar{N}_{1,2}(r,a) = \bar{N}(r, \frac{1}{f_1 - a}) + \bar{N}(r, \frac{1}{f_2 - a}) - 2\bar{N}_0(r,a)$.

Also let $\overline{n}_0^{(k)}(r,a)$, $\bar{N}_{1,2}^{(k)}(r,a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$ and $f_2^{(k)}$ where k is any non-negative integer.

Now for any integer $n > 1$ we set the following quantities :

$$_{(n)}\delta_{1,2}(a) = 1 - \limsup_{r\to\infty} \frac{N_{1,2}(r,a)}{T(r, f_1^{(n)}) + T(r, f_2^{(n)})},$$

$$_{(n)}\delta_{1,2}^{(k)}(a) = 1 - \limsup_{r\to\infty} \frac{N_{1,2}^{(k)}(r,a)}{T(r, f_1^{(n)}) + T(r, f_2^{(n)})},$$

$$_{(n)}\delta_{0}(a) = 1 - \limsup_{r\to\infty} \frac{N_{0}(r,a)}{T(r, f_1^{(n)}) + T(r, f_2^{(n)})},$$

$$_{(n)}\Theta_{1,2}(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_{1,2}(r,a)}{T(r, f_1^{(n)}) + T(r, f_2^{(n)})},$$

$$_{(n)}\Theta_{1,2}^{(k)}(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_{1,2}^{(k)}(r,a)}{T(r, f_1^{(n)}) + T(r, f_2^{(n)})} \text{ and}$$

$$_{(n)}\Theta_{0}(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_{0}(r,a)}{T(r, f_1^{(n)}) + T(r, f_2^{(n)})}.$$

In our paper we intend to estimate the ratio $\chi_n(r) = \frac{T(r,f_1)+T(r,f_2)}{T(r,f_1^{(n)})+T(r,f_2^{(n)})}$ as r tending to infinity where n is any integer greater than 1.

We do not explain the definitions and standard notations of Nevanlinna theory because those are available in [1]. The term $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. (see [3]) *Let f be a meromorphic function of finite order such that $\sum_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$. Then for any non-negative integer k,*

$$\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}=1.$$

The following lemma is due to Milloux, see [1], p. 55.

Lemma 2. (see [1]) *Let k be any positive integer and $\Psi=\sum_{i=0}^{k}a_i f^{(i)}$ where a_i are meromorphic functions such that $T(r,a_i)=S(r,f)$. Then*

$$m(r,\frac{\Psi}{f})=S(r,f).$$

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f_1 and f_2 be any two meromorphic functions of finite order such that $\bar{N}(r,f_1)=S(r,f_1)$ and $\bar{N}(r,f_2)=S(r,f_2)$. Also let a be a finite non-zero complex number. Then for any two positive integers n and k with $n>1$,*

$$_{(n)}\delta_{1,2}(0)+_{(n)}\delta_{1,2}(a)+2_{(n)}\delta_0(a)+2_{(n)}\delta_0(a)+\limsup_{r\to\infty}\chi_n(r)\leq 6.$$

Proof. Let us consider the following identity

$$\frac{a}{f}=1-\frac{f-a}{f^{(k)}}.\frac{f^{(k)}}{f}.$$

Since $m(r,\frac{1}{f})\leq m(r,\frac{a}{f})+O(1)$, we get from the above identity,

$$m(r,\frac{1}{f})\leq m(r,\frac{f-a}{f^{(k)}})+S(r,f). \tag{1}$$

Now by Nevanlinna's First Fundamental Theorem and Milloux's Theorem (see [1], p. 55), it follows from (1) that

$$m(r,\frac{1}{f})\leq T(r,\frac{f-a}{f^{(k)}})-N(r,\frac{f-a}{f^{(k)}})+S(r,f)\,,$$

$$m(r,\frac{1}{f})\le T(r,\frac{f^{(k)}}{f-a})-N(r,\frac{f-a}{f^{(k)}})+S(r,f)\,,$$

$$m(r,\frac{1}{f})\le N(r,\frac{f^{(k)}}{f-a})-N(r,\frac{f-a}{f^{(k)}})+S(r,f)\,. \tag{2}$$

In view of [1], p. 34, and as $N(r,\frac{1}{f^{(k)}})\ge 0$, it follows from (2) that

$$m(r,\frac{1}{f})\le N(r,f^{(k)})+N(r,\frac{1}{f-a})-N(r,f-a)-N(r,\frac{1}{f^{(k)}})+S(r,f)\,,$$

$$m(r,\frac{1}{f})\le N(r,f)+k\bar{N}(r,f)+N(r,\frac{1}{f-a})-N(r,f)-N(r,\frac{1}{f^{(k)}})+S(r,f)\,,$$

$$T(r,f)\le k\bar{N}(r,f)+N(r,\frac{1}{f})+N(r,\frac{1}{f-a})+S(r,f). \tag{3}$$

Applying this inequality for f_1 and f_2 we get that

$$\{T(r,f_1)+T(r,f_2)\}\le k\{\bar{N}(r,f_1)+\bar{N}(r,f_2)\}+\{N(r,\frac{1}{f_1})$$
$$+N(r,\frac{1}{f_2})\}+\{N(r,\frac{1}{f_1-a})+N(r,\frac{1}{f_2-a})\}+S(r,f_1)+S(r,f_2). \tag{4}$$

In view of $\bar{N}(r,f_1)=S(r,f_1)$, $\bar{N}(r,f_2)=S(r,f_2)$ and dividing both sides of (4) by $\{T(r,f_1^{(n)})+T(r,f_2^{(n)})\}$ and taking limit superior we obtain that

$$\limsup_{r\to\infty}\frac{T(r,f_1)+T(r,f_2)}{T(r,f_1^{(n)})+T(r,f_2^{(n)})}\le\{1-{}_{(n)}\,\delta_{1,2}(0)\}+2\{1-{}_{(n)}\,\delta_0(0)\}$$
$$+\{1-{}_{(n)}\,\delta_{1,2}(a)\}+2\{1-{}_{(n)}\,\delta_0(a)\}$$
$${}_{(n)}\delta_{1,2}(0)+{}_{(n)}\,\delta_{1,2}(a)+2_{(n)}\delta_0(a)+2_{(n)}\delta_0(a)+\limsup_{r\to\infty}\chi_n(r)\le 6.$$

This proves the theorem. □

Theorem 2. *Let f_1 and f_2 be any two meromorphic functions of finite order such that $T(r,f_1^{(k)})\sim lT(r,f_1)$ and $T(r,f_2^{(k)})\sim lT(r,f_2)$, where k is any positive integer and $l\ge 1$. Then for any integer $n>1$,*

$$\frac{(l-1)}{k}\limsup_{r\to\infty}\chi_n(r)+{}_{(n)}\,\delta_{1,2}(\infty)+2_{(n)}\delta_0(\infty)\le 3.$$

Proof. In view of Milloux's Theorem (see [1], p. 55) we obtain that

$$T(r,f^{(k)})=N(r,f^{(k)})+m(r,f^{(k)})=N(r,f)+k\bar{N}(r,f)+m(r,f)+S(r,f)$$
$$=T(r,f)+k\bar{N}(r,f)+S(r,f)\le T(r,f)+kN(r,f)+S(r,f). \tag{5}$$

Applying (5) on f_1 and f_2 it follows that

$$T(r,f_1^{(k)})+T(r,f_2^{(k)})\le\{T(r,f_1)+T(r,f_2)\}\\ +k\{N(r,f_1)+N(r,f_2)\}+S(r,f_1)+S(r,f_2). \quad (6)$$

Since $T(r,f_i^{(k)})\sim lT(r,f_i)$ for $i=1,2$ we get from (6) that

$$(l-1)\{T(r,f_1)+T(r,f_2)\}\le k\{N(r,f_1)+N(r,f_2)\}+S(r,f_1)+S(r,f_2). \quad (7)$$

Now dividing both sides of (7) by $\{T(r,f_1^{(n)})+T(r,f_2^{(n)})\}$ and taking limit superior we obtain that

$$(l-1)\{\limsup_{r\to\infty}\frac{T(r,f_1)+T(r,f_2)}{T(r,f_1^{(n)})+T(r,f_2^{(n)})}\}\le k[\{1-_{(n)}\delta_{1,2}(\infty)\}+2\{1-_{(n)}\delta_0(\infty)\}]\,,$$

$$\frac{(l-1)}{k}\limsup_{r\to\infty}\chi_n(r)+_{(n)}\delta_{1,2}(\infty)+2_{(n)}\delta_0(\infty)\le 3. \quad (8)$$

Thus the theorem follows from (8). □

Theorem 3. *Let f_1 and f_2 be two meromorphic functions of finite order with $\bar{N}(r,f_1)=S(r,f_1)$ and $\bar{N}(r,f_2)=S(r,f_2)$. Then for any positive integer k and $n>1$ and for any two distinct finite complex numbers a and b*

$$2_{(n)}\delta_{1,2}(a)+_{(n)}\delta_{1,2}^{(k)}(b)+4_{(n)}\delta_0(a)+2_{(n)}\delta_0(b)+\limsup_{r\to\infty}\chi_n(r)\le 9.$$

Proof. Considering the identity

$$\frac{b-a}{f-a}=\frac{f^{(k)}}{f-a}\{\frac{f-a}{f^{(k)}}-\frac{f-b}{f^{(k)}}\}$$

we obtain in view of Milloux's Theorem (see [1], p. 55)

$$m(r,\frac{b-a}{f-a})\le m(r,\frac{f-a}{f^{(k)}})+m(r,\frac{f-b}{f^{(k)}})+S(r,f)\,,$$

$$m(r,\frac{b-a}{f-a}) \le T(r,\frac{f-a}{f^{(k)}})-N(r,\frac{f-a}{f^{(k)}})\\ +T(r,\frac{f-b}{f^{(k)}})-N(r,\frac{f-b}{f^{(k)}})+S(r,f). \quad (9)$$

Since $m(r,\frac{1}{f-a})\le m(r,\frac{b-a}{f-a})+O(1)$ and $T(r,f)=T(r,\frac{1}{f})+O(1)$, it follows from (9) that,

$$m(r,\frac{1}{f-a}) \le N(r,\frac{f^{(k)}}{f-a})-N(r,\frac{f-a}{f^{(k)}})+N(r,\frac{f^{(k)}}{f-b})\\ -N(r,\frac{f-b}{f^{(k)}})+S(r,f). \quad (10)$$

In view of [1], p. 34, and as $N(r, \frac{1}{f^{(k)}}) \geq 0$ we get from (10) that

$$\begin{aligned} m(r, \frac{1}{f-a}) \;\leq\; & N(r, f^{(k)}) + N(r, \frac{1}{f-a}) - N(r, f-a) \\ & - N(r, \frac{1}{f^{(k)}}) + N(r, f^{(k)}) + N(r, \frac{1}{f-b}) - N(r, f-b) \\ & - N(r, \frac{1}{f^{(k)}}) + S(r, f) \end{aligned}$$

$$= 2N(r, f^{(k)}) - 2N(r, f) - 2N(r, \frac{1}{f^{(k)}}) + N(r, \frac{1}{f-a}) + N(r, \frac{1}{f-b}) + S(r, f),$$

$$T(r, f) \leq 2k\bar{N}(r, f) + 2N(r, \frac{1}{f-a}) + N(r, \frac{1}{f-b}) + S(r, f). \tag{11}$$

Now applying (11) for f_1 and f_2 we obtain that

$$\begin{aligned} \{T(r, f_1) + T(r, f_2)\} \;\leq\; & 2k\{\bar{N}(r, f_1) + \bar{N}(r, f_2)\} \\ & + 2\{N(r, \frac{1}{f_1 - a}) + N(r, \frac{1}{f_2 - a})\} \\ & + \{N(r, \frac{1}{f_1 - b}) + N(r, \frac{1}{f_2 - b})\} \\ & + S(r, f_1) + S(r, f_2). \end{aligned} \tag{12}$$

As $\bar{N}(r, f_i) = S(r, f_i)$ for $i = 1, 2$; dividing both sides of (12) by $\{T(r, f_1^{(n)}) + T(r, f_2^{(n)})\}$ and taking limit superior we get that

$$\begin{aligned} \limsup_{r\to\infty} \frac{T(r, f_1) + T(r, f_2)}{T(r, f_1^{(n)}) + T(r, f_2^{(n)})} \;\leq\; & 2\{1 -_{(n)} \delta_{1,2}(a)\} + 4\{1 -_{(n)} \delta_0(a)\} \\ & + \{1 -_{(n)} \delta_{1,2}^{(k)}(b)\} + 2\{1 -_{(n)} \delta_0(b)\}, \end{aligned}$$

$$2_{(n)}\delta_{1,2}(a) +_{(n)} \delta_{1,2}^{(k)}(b) + 4_{(n)}\delta_0(a) + 2_{(n)}\delta_0(b) + \limsup_{r\to\infty} \chi_n(r) \leq 9.$$

Thus the theorem is proved. □

Theorem 4. *Let f_1 and f_2 be any two meromorphic functions both of finite order such that $\bar{N}(r, f_1) = S(r, f_1)$ and $\bar{N}(r, f_2) = S(r, f_2)$. Then for every integer k and for every integer $p > 1$,*

$${}_{(p)}\delta_{1,2}^{(k)}(a) +_{(p)} \delta_{1,2}(a) + 2_{(p)}\delta_0^{(k)}(a) + 2_{(p)}\delta_0(a) + \limsup_{r\to\infty} \chi_p(r) \leq 6.$$

Proof. From the identity

$$\frac{1}{f-a}=\frac{1}{a}\{\frac{f^{(k)}}{f-a}-\frac{f^{(k)}-a}{f^{(n)}}.\frac{f^{(n)}}{f-a}\},$$

where $0\le k<n$ and by Milloux's Theorem [1], p. 55, we get that

$$m(r,\frac{1}{f-a})\le m(r,\frac{f^{(k)}-a}{f^{(n)}})+S(r,f). \tag{13}$$

Now by Nevanlinna's First Fundamental Theorem it follows from (13) that

$$m(r,\frac{1}{f-a})\le T(r,\frac{f^{(k)}-a}{f^{(n)}})-N(r,\frac{f^{(k)}-a}{f^{(n)}})+S(r,f),$$

$$m(r,\frac{1}{f-a})\le T(r,\frac{f^{(n)}}{f^{(k)}-a})-N(r,\frac{f^{(k)}-a}{f^{(n)}})+S(r,f),$$

$$m(r,\frac{1}{f-a})\le N(r,\frac{f^{(n)}}{f^{(k)}-a})-N(r,\frac{f^{(k)}-a}{f^{(n)}})+S(r,f). \tag{14}$$

Now in view of [1], p. 34, and as $N(r,\frac{1}{f^{(n)}})\ge 0$ we obtain from (14) that

$$m(r,\frac{1}{f-a})\le N(r,f^{(n)})+N(r,\frac{1}{f^{(k)}-a})-N(r,f^{(k)}-a)-N(r,\frac{1}{f^{(n)}})+S(r,f),$$

$$\begin{aligned}T(r,\frac{1}{f-a}) &\le \{N(r,f)+n\bar{N}(r,f)\}\\ &\quad +N(r,\frac{1}{f^{(k)}-a})+N(r,\frac{1}{f-a})\\ &\quad -\{N(r,f)+k\bar{N}(r,f)\}+S(r,f),\end{aligned}$$

$$\begin{aligned}T(r,f) &\le (n-k)\bar{N}(r,f)+N(r,\frac{1}{f^{(k)}-a})\\ &\quad +N(r,\frac{1}{f-a})+S(r,f).\end{aligned} \tag{15}$$

Applying this inequality for f_1 and f_2 it follows from (15) that

$$\begin{aligned}T(r,f_1)+T(r,f_2) &\le (n-k)\{\bar{N}(r,f_1)+\bar{N}(r,f_2)\}\\ &\quad +N(r,\frac{1}{f_1^{(k)}-a})+N(r,\frac{1}{f_2^{(k)}-a})\\ &\quad +N(r,\frac{1}{f_1-a})+N(r,\frac{1}{f_2-a})\\ &\quad +S(r,f_1)+S(r,f_2).\end{aligned} \tag{16}$$

As $\bar{N}(r,f_1)=S(r,f_1)$ and $\bar{N}(r,f_2)=S(r,f_2)$, dividing both sides of (16) by

$\{T(r,f_1^{(p)})+T(r,f_2^{(p)})\}$ and taking limit superior we get that

$$\limsup_{r\to\infty}\frac{T(r,f_1)+T(r,f_2)}{T(r,f_1^{(p)})+T(r,f_2^{(p)})} \leq \{1-{}_{(p)}\,\delta_{1,2}^{(k)}(a)\}+2\{1-{}_{(p)}\,\delta_0^{(k)}(a)\} +\{1-{}_{(p)}\,\delta_{1,2}(a)\}+2\{1-{}_{(p)}\,\delta_0(a)\}\,,$$

$$_{(p)}\delta_{1,2}^{(k)}(a)+{}_{(p)}\,\delta_{1,2}(a)+2_{(p)}\delta_0^{(k)}(a)+2_{(p)}\delta_0(a)+\limsup_{r\to\infty}\chi_p(r)\leq 6.$$

This proves the theorem. □

Theorem 5. *Let f_1 and f_2 be any two meromorphic functions of finite order such that $\sum\limits_{\alpha\neq\infty}\delta(\alpha;f_1)=\delta(\infty;f_1)=1$ and $\sum\limits_{\alpha\neq\infty}\delta(\alpha;f_2)=\delta(\infty;f_2)=1$. Also, let a be a finite complex number and b,c be two distinct non-zero complex numbers. Then for any two positive integers k and $n>1$*

$$_{(n)}\delta_{1,2}(a)+{}_{(n)}^{(k)}\,\Theta_{1,2}(b)+{}_{(n)}^{(k)}\,\Theta_{1,2}(c)+2_{(n)}\delta_0(a)+2_{(n)}^{(k)}\Theta_0(b) +2_{(n)}^{(k)}\Theta_0(c)+\limsup_{r\to\infty}\chi_n(r)\leq 9.$$

Proof. Since $\frac{1}{f-a}=\frac{f^{(k)}}{f-a}\cdot\frac{1}{f^{(k)}}$ by Milloux's Theorem [1], p. 55, we obtain that

$$m(r,\frac{1}{f-a})\leq m(r,\frac{1}{f^{(k)}})+S(r,f). \tag{17}$$

Applying Nevanlinna's First Fundamental Theorem we get from (17) that

$$m(r,\frac{1}{f-a})\leq T(r,f^{(k)})-N(r,\frac{1}{f^{(k)}})+S(r,f). \tag{18}$$

Now by Nevanlinna's Second Fundamental Theorem and Lemma 1 it follows from (18) that

$$m(r,\frac{1}{f-a}) \leq \bar{N}(r,\frac{1}{f^{(k)}})+\bar{N}(r,\frac{1}{f^{(k)}-b}) +\bar{N}(r,\frac{1}{f^{(k)}-c})-N(r,\frac{1}{f^{(k)}})+S(r,f). \tag{19}$$

Since $\bar{N}(r,\frac{1}{f^{(k)}})-N(r,\frac{1}{f^{(k)}})\leq 0$, we obtain from (19) that

$$m(r,\frac{1}{f-a})\leq\bar{N}(r,\frac{1}{f^{(k)}-b})+\bar{N}(r,\frac{1}{f^{(k)}-c})+S(r,f)\,,$$

$$T(r,f) \leq N(r,\frac{1}{f-a})+\bar{N}(r,\frac{1}{f^{(k)}-b})$$

$$+\bar{N}(r,\frac{1}{f^{(k)}-c})+S(r,f). \tag{20}$$

Now applying (20) for f_1 and f_2 it follows that,

$$\begin{aligned}\{T(r,f_1)+T(r,f_2)\} \quad \leq \quad & \{N(r,\frac{1}{f_1-a})+N(r,\frac{1}{f_2-a})\} \\ & +\{\bar{N}(r,\frac{1}{f_1^{(k)}-b})+\bar{N}(r,\frac{1}{f_2^{(k)}-b})\} \\ & +\{\bar{N}(r,\frac{1}{f_1^{(k)}-c})+\bar{N}(r,\frac{1}{f_2^{(k)}-c})\} \\ & +S(r,f_1)+S(r,f_2). \end{aligned} \tag{21}$$

Dividing both sides of (21) by $\{T(r,f_1^{(n)})+T(r,f_2^{(n)})\}$ and taking limit superior we get that

$$\begin{aligned}\limsup_{r\to\infty}\frac{T(r,f_1)+T(r,f_2)}{T(r,f_1^{(n)})+T(r,f_2^{(n)})} \quad \leq \quad & \{1-_{(n)}\delta_{1,2}(a)\}+2\{1-_{(n)}\delta_0(a)\} \\ & +\{1-^{(k)}_{(n)}\Theta_{1,2}(b)\}+2\{1-^{(k)}_{(n)}\Theta_0(b)\} \\ & +\{1-^{(k)}_{(n)}\Theta_{1,2}(c)\}+2\{1-^{(k)}_{(n)}\Theta_0(c)\}, \end{aligned}$$

$$\begin{aligned} {}_{(n)}\delta_{1,2}(a)+^{(k)}_{(n)}\Theta_{1,2}(b)+^{(k)}_{(n)}\Theta_{1,2}(c)+2_{(n)}\delta_0(a)+2^{(k)}_{(n)}\Theta_0(b) \\ +2^{(k)}_{(n)}\Theta_0(c)+\limsup_{r\to\infty}\chi_n(r)\leq 9. \end{aligned}$$

Thus the theorem is established. □

Theorem 6. *Let k be any positive integer and a be a finite complex number. Then for any two integers k and $n>1$ and for any two meromorphic functions f_1, f_2 of finite orders with $\bar{N}(r,f_i)=S(r,f_i)$ and $T(r,f_i^{(k)})\sim lT(r,f_i)$ for $i=1,2$*

$${}_{(n)}\delta_{1,2}(0)+2_{(n)}\delta_0(0)+l\limsup_{r\to\infty}\chi_n(r)\leq 3,$$

where $l\geq 1$.

Proof. Let $b\neq a$ be a finite complex number. Since

$$\frac{a-b}{f^{(k)}-a}=\frac{f}{f^{(k)}-a}\{\frac{f^{(k)}-b}{f}-\frac{f^{(k)}-a}{f}\},$$

we obtain in view of Milloux's Theorem [1], p. 55, and Nevanlinna's First

Fundamental Theorem

$$m(r,\frac{a-b}{f^{(k)}-a})\le m(r,\frac{f}{f^{(k)}-a})+S(r,f)\,,$$

$$m(r,\frac{1}{f^{(k)}-a})\le T(r,\frac{f}{f^{(k)}-a})-N(r,\frac{f}{f^{(k)}-a})+S(r,f)\,,$$

$$m(r,\frac{1}{f^{(k)}-a})\le T(r,\frac{f^{(k)}-a}{f})-N(r,\frac{f}{f^{(k)}-a})+S(r,f)\,,$$

$$m(r,\frac{1}{f^{(k)}-a})\le N(r,\frac{f^{(k)}-a}{f})-N(r,\frac{f}{f^{(k)}-a})+S(r,f). \tag{22}$$

In view of [1], p. 34, it follows from (22) that

$$\begin{aligned} m(r,\frac{1}{f^{(k)}-a}) &\le N(r,f^{(k)}-a)+N(r,\frac{1}{f})-N(r,f) \\ &\quad -N(r,\frac{1}{f^{(k)}-a})+S(r,f)\,, \\ T(r,\frac{1}{f^{(k)}-a}) &\le \{N(r,f)+k\bar{N}(r,f)\}+N(r,\frac{1}{f}) \\ &\quad -N(r,f)+S(r,f)\,, \end{aligned}$$

$$T(r,f^{(k)})\le k\bar{N}(r,f)+N(r,\frac{1}{f})+S(r,f). \tag{23}$$

Since $\bar{N}(r,f_i)=S(r,f_i)$ and $T(r,f_i^{(k)})\sim lT(r,f_i)$ for $i=1,2$; applying (23) for f_1 and f_2 it follows that,

$$l\{T(r,f_1)+T(r,f_2)\}\le N(r,\frac{1}{f_1})+N(r,\frac{1}{f_2})+S(r,f_1)+S(r,f_2). \tag{24}$$

Now dividing both sides of (24) by $\{T(r,f_1^{(n)})+T(r,f_2^{(n)})\}$ and taking limit superior it follows that

$$l\ \limsup_{r\to\infty}\frac{T(r,f_1)+T(r,f_2)}{T(r,f_1^{(n)})+T(r,f_2^{(n)})}\le\{1-{}_{(n)}\,\delta_{1,2}(0)\}+2\{1-{}_{(n)}\,\delta_0(0)\}\,,$$

$${}_{(n)}\delta_{1,2}(0)+2{}_{(n)}\delta_0(0)+l\ \limsup_{r\to\infty}\chi_n(r)\le 3.$$

This proves the theorem. □

Theorem 7. *Let f_1 and f_2 be any two meromorphic functions of finite order with $\bar{N}(r,f_1)=S(r,f_1)$ and $\bar{N}(r,f_2)=S(r,f_2)$ and α be a non-zero finite* *complex number. Then for any two positive integers k and $n>1$*

$$_{(n)}\delta_{1,2}^{(k)}(\alpha)+_{(n)}\delta_{1,2}(0)+2_{(n)}\delta_0^{(k)}(\alpha)+2_{(n)}\delta_0(0)+\limsup_{r\to\infty}\chi_n(r)\leq 6.$$

Proof. Considering the identity

$$\frac{\alpha}{f}=\frac{f^{(k)}}{f}-\frac{f^{(k)}-\alpha}{f^{(k+1)}}.\frac{f^{(k+1)}}{f},$$

we get in view of Milloux's Theorem [1], p. 55, and Nevanlinna's First Fundamental Theorem

$$m(r,\frac{1}{f})\leq m(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}})+S(r,f)\,,$$

$$m(r,\frac{1}{f})\leq T(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}})-N(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}})+S(r,f)\,,$$

$$m(r,\frac{1}{f})\leq T(r,\frac{f^{(k+1)}}{f^{(k)}-\alpha})-N(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}})+S(r,f)\,,$$

$$m(r,\frac{1}{f})\leq N(r,\frac{f^{(k+1)}}{f^{(k)}-\alpha})-N(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}})+S(r,f). \tag{25}$$

Now in view of [1], p. 34, and as $N(r,\frac{1}{f^{(k+1)}})\geq 0$ it follows from (25) that

$$\begin{aligned} m(r,\frac{1}{f}) \;&\leq\; N(r,f^{(k+1)})+N(r,\frac{1}{f^{(k)}-\alpha})-N(r,f^{(k)}-\alpha)\\ &\quad -N(r,\frac{1}{f^{(k+1)}})+S(r,f)\,,\end{aligned}$$

$$m(r,\frac{1}{f})\leq N(r,\frac{1}{f^{(k)}-\alpha})+N(r,f^{(k+1)})-N(r,f^{(k)})+S(r,f)\,,$$

$$m(r,\frac{1}{f})\leq N(r,\frac{1}{f^{(k)}-\alpha})+\bar{N}(r,f)+S(r,f)\,,$$

$$T(r,f)\leq N(r,\frac{1}{f^{(k)}-\alpha})+N(r,\frac{1}{f})+\bar{N}(r,f)+S(r,f). \tag{26}$$

Applying (26) for f_1 and f_2 we obtain that

$$\begin{aligned}\{T(r,f_1)+T(r,f_2)\} \;&\leq\; N(r,\frac{1}{f_1^{(k)}-\alpha})+N(r,\frac{1}{f_2^{(k)}-\alpha})\\ &\quad +N(r,\frac{1}{f_1})+N(r,\frac{1}{f_2})+\bar{N}(r,f_1)\\ &\quad +\bar{N}(r,f_2)+S(r,f_1)+S(r,f_2).\end{aligned} \tag{27}$$

As $\bar{N}(r, f_i) = S(r, f_i)$ for $i = 1, 2$; dividing both sides of (27) by $\{T(r, f_1^{(n)}) + T(r, f_2^{(n)})\}$ and taking limit superior we get that

$$\limsup_{r\to\infty} \frac{T(r, f_1) + T(r, f_2)}{T(r, f_1^{(n)}) + T(r, f_2^{(n)})} \leq \{1 -_{(n)} \delta_{1,2}^{(k)}(\alpha)\} + 2\{1 -_{(n)} \delta_0^{(k)}(\alpha)\} + \{1 -_{(n)} \delta_{1,2}(0)\} + 2\{1 -_{(n)} \delta_0(0)\},$$

$$_{(n)}\delta_{1,2}^{(k)}(\alpha) +_{(n)} \delta_{1,2}(0) + 2_{(n)}\delta_0^{(k)}(\alpha) + 2_{(n)}\delta_0(0) + \limsup_{r\to\infty} \chi_n(r) \leq 6.$$

Thus the theorem is established. □

References

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