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ON ANTI FUZZY BI-IDEALS IN RINGS

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Abstract: The purpose of this paper is to introduce the concept of anti fuzzy bi-ideals in rings and to study some of their properties.

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1. Introduction, Definitions and Preliminaries

Zadeh [10] coined the idea of a fuzzy subset A of a set X as a function from X into $[0,1]$. Using this concept Rosenfeld [9] developed some results in fuzzy group theory. Later Kuroki [3] introduced the notion of fuzzy ideals in semi groups and Liu [6] studied them in rings. Lajos and Szasz [5] initiated the idea of bi-ideals in a ring. Kandasamy [4] and Majumder and Sardar [7] respectively used the concept of fuzzy translation and fuzzy magnified translation in fuzzy group theory. Majumder and Sardar [8] also explored the idea of anti fuzzy bi-ideals in fuzzy group theory. In this paper we develop our results on anti fuzzy bi-ideals in rings. We also intend to establish a few results on fuzzy magnified translation of anti fuzzy bi-ideals in rings.

We now review some definitions that are used in this paper.

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Definition 1. (see [9]) Let μ_i, $i \in I$ be fuzzy subsets of a ring R. The intersection of the fuzzy sets μ_i is defined as follows:

$$[\cap \mu_i](x) = \inf_{i \in I} [\mu_i(x)], \quad x \in X.$$

Definition 2. (see [9]) Let μ be a fuzzy subset of a ring R. Imμ is defined as

$$\text{Im}\mu = \{t \in [0,1] \mid \mu(x) = t \text{ for some } x \in X\}.$$

Definition 3. (see [5]) A sub ring S of a ring R is called a bi-ideal of R if $SRS \subseteq S$ holds where SRS is the additive subgroup of R generated by the set of all elements of the form srs, $s \in S$ and $r \in R$.

Definition 4. (see [9]) A fuzzy subset μ of a ring R is called an anti fuzzy left (right) ideal of R, if for every $x, y \in R$,

(i) μ is an anti fuzzy subgroup of $(R, +)$, i.e.,

$$\mu(x - y) \leq \max\{\mu(x), \mu(y)\} \text{ and}$$

(ii) $\mu(xy) \leq \mu(y)\,(\mu(xy) \leq \mu(x))$.

If μ is both an anti fuzzy left ideal and an anti fuzzy right ideal of R, then it is called an anti fuzzy ideal of R.

Definition 5. (see [1]) A non-empty fuzzy subset μ of a ring R (i.e. $\mu(x) \neq 0$ for some $x \in R$) is called a fuzzy bi-ideal of R if:

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, and

(iii) $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in R$.

Example 1. Let R be the ring of all 2×2 matrices over the ring of integers with respect to the matrix addition and multiplication. Let μ be a fuzzy subset of R defined as follows:

$$\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 1 \text{ if } a = b = c = d = 0$$

$$= \frac{1}{2} \text{ if } a \text{ is a non-zero even integer and } b = c = d = 0$$

$$= \frac{1}{3} \text{ if } a \text{ is a non-zero odd integer and } b = c = d = 0$$

$$= 0 \text{ in all other cases.}$$

Then μ is a fuzzy bi-ideal of R.

Definition 6. A non-empty fuzzy subset μ of a ring R (i.e. $\mu(x) \neq 0$ for some $x \in R$) is called an anti fuzzy bi-ideal of R if:

(i) $\mu(x-y) \leq \max\{\mu(x), \mu(y)\}$,

(ii) $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$, and

(iii) $\mu(xyz) \leq \max\{\mu(x), \mu(z)\}$ for all $x, y, z \in R$.

Definition 7. (see [2]) Let μ be a non-empty fuzzy subset of a ring R (i.e. $\mu(x) \neq 0$ for some $x \in R$) and also let $\alpha \in [0, 1 - \sup\{\mu(x) : x \in R\}]$, $\beta \in [0,1]$. Then the fuzzy magnified translation $\mu^c_{\beta\alpha}$ of μ in R is defined as

$$\mu^c_{\beta\alpha}(x) = \beta.\mu(x) + \alpha \text{ for all } x \in R.$$

It is also a fuzzy subset of R.

In particular if $\beta = 1$ then μ^T_α is called the fuzzy translation of μ, i.e.,

$$\mu^T_\alpha(x) = \mu(x) + \alpha \text{ for all } x \in R.$$

Also when $\alpha = 0$ then μ^M_β is called the fuzzy multiplication of μ, i.e.,

$$\mu^M_\beta(x) = \beta\mu(x) \text{ for all } x \in R.$$

2. Theorems

In this section we present the main results of our paper.

Theorem 1. *Every anti fuzzy left (right, two-sided) ideal of a ring R is an anti fuzzy bi-ideal of R.*

Proof. Let μ be an anti fuzzy left ideal of R. Then for all $x, y \in R$

$$\mu(x-y) \leq \max\{\mu(x), \mu(y)\}$$

and

$$\mu(xy) \leq \max\{\mu(x), \mu(y)\}.$$

Again let $x, y, z \in R$. Now

$$\begin{aligned} \mu(xyz) &= \mu((xy)z) \leq \mu(z) \\ &\leq \max\{\mu(x), \mu(z)\}. \end{aligned}$$

So μ is an anti fuzzy bi-ideal of R.

Similarly we can prove the other statements. □

Theorem 2. *Let μ be a non empty fuzzy sub set of a ring R. Then μ is an anti fuzzy bi-ideal of R iff the fuzzy magnified translation $\mu^c_{\beta\alpha}$ of μ is an anti fuzzy bi-ideal of R.*

Proof. Let μ be an anti fuzzy bi-ideal of a ring R.

Now for all $x, y, z \in R$,

$$\mu^c_{\beta\alpha}(x-y) = \beta.\mu(x-y) + \alpha \leq \beta \max\{\mu(x), \mu(y)\} + \alpha$$
$$= \max\{\beta.\mu(x) + \alpha, \beta.\mu(y) + \alpha\},$$
$$\mu^c_{\beta\alpha}(x-y) \leq \max\left\{\mu^c_{\beta\alpha}(x), \mu^c_{\beta\alpha}(y)\right\}.$$

Again

$$\mu^c_{\beta\alpha}(xy) = \beta.\mu(xy) + \alpha \leq \beta \max\{\mu(x), \mu(y)\} + \alpha$$
$$= \max\{\beta.\mu(x) + \alpha, \beta.\mu(y) + \alpha\},$$
$$\mu^c_{\beta\alpha}(xy) \leq \max\left\{\mu^c_{\beta\alpha}(x), \mu^c_{\beta\alpha}(y)\right\}.$$

Also

$$\mu^c_{\beta\alpha}(xyz) = \beta.\mu(xyz) + \alpha \leq \beta \max\{\mu(x), \mu(z)\} + \alpha$$
$$= \max\{\beta.\mu(x) + \alpha, \beta.\mu(z) + \alpha\},$$
$$\mu^c_{\beta\alpha}(xyz) \leq \max\left\{\mu^c_{\beta\alpha}(x), \mu^c_{\beta\alpha}(z)\right\}.$$

Thus $\mu^c_{\beta\alpha}$ is an anti fuzzy bi-ideal of R.

Conversely let $\mu^c_{\beta\alpha}$ be an anti fuzzy bi-ideal of R. Then for all $x, y, z \in R$,

$$\mu(x-y) = \frac{1}{\beta}\left[\mu^c_{\beta\alpha}(x-y) - \alpha\right] \leq \frac{1}{\beta}\left[\max\left\{\mu^c_{\beta\alpha}(x), \mu^c_{\beta\alpha}(y)\right\} - \alpha\right]$$
$$= \frac{1}{\beta}\left[\max\left\{\mu^c_{\beta\alpha}(x) - \alpha, \mu^c_{\beta\alpha}(y) - \alpha\right\}\right] = \max\left\{\frac{\mu^c_{\beta\alpha}(x) - \alpha}{\beta}, \frac{\mu^c_{\beta\alpha}(y) - \alpha}{\beta}\right\},$$
$$\mu(x-y) \leq \max\{\mu(x), \mu(y)\}.$$

Again

$$\mu(xy) = \frac{1}{\beta}\left[\mu^c_{\beta\alpha}(xy) - \alpha\right] \leq \frac{1}{\beta}\left[\max\left\{\mu^c_{\beta\alpha}(x), \mu^c_{\beta\alpha}(y)\right\} - \alpha\right]$$
$$= \frac{1}{\beta}\left[\max\left\{\mu^c_{\beta\alpha}(x) - \alpha, \mu^c_{\beta\alpha}(y) - \alpha\right\}\right] = \max\left\{\frac{\mu^c_{\beta\alpha}(x) - \alpha}{\beta}, \frac{\mu^c_{\beta\alpha}(y) - \alpha}{\beta}\right\},$$
$$\mu(xy) \leq \max\{\mu(x), \mu(y)\}.$$

Also

$$\mu(xyz) = \frac{1}{\beta}\left[\mu^c_{\beta\alpha}(xyz) - \alpha\right] \leq \frac{1}{\beta}\left[\max\left\{\mu^c_{\beta\alpha}(x), \mu^c_{\beta\alpha}(z)\right\} - \alpha\right]$$

$$= \frac{1}{\beta}\left[\max\left\{\mu^c_{\beta\alpha}(x) - \alpha, \mu^c_{\beta\alpha}(z) - \alpha\right\}\right] = \max\left\{\frac{\mu^c_{\beta\alpha}(x) - \alpha}{\beta}, \frac{\mu^c_{\beta\alpha}(z) - \alpha}{\beta}\right\},$$

$$\mu(xyz) \leq \max\{\mu(x), \mu(z)\}.$$

Hence μ is an anti fuzzy bi-ideal of R. □

Corollary 1. *Let μ be a non empty fuzzy sub set of a ring R. Then μ is an anti fuzzy bi-ideal of R iff the fuzzy translation μ^T_α of μ is an anti fuzzy bi-ideal of R.*

Proof. Taking $\beta = 1$ the above corollary follows from Theorem 2. □

Corollary 2. *Let μ be a non empty fuzzy sub set of a ring R. Then μ is an anti fuzzy bi-ideal of R iff the fuzzy multiplication μ^M_β of μ is an anti fuzzy bi-ideal of R.*

Proof. Taking $\alpha = 0$ Corollary 2 follows from Theorem 2. □

Theorem 3. *If μ be an anti fuzzy left (right, two-sided) ideal of a ring R then the fuzzy magnified translation $\mu^c_{\beta\alpha}$ of μ is an anti fuzzy bi-ideal of R.*

Proof. The proof of Theorem 3 directly follows from Theorem 1 and Theorem 2. □

In the line of Theorem 3 we may state the following two corollaries without proof.

Corollary 3. *Let μ be an anti fuzzy left (right, two-sided) ideal of a ring R. Then the fuzzy translation μ^T_α of μ is an anti fuzzy bi-ideal of R.*

Corollary 4. *Let μ be an anti fuzzy left (right, two-sided) ideal of a ring R. Then the fuzzy multiplication μ^M_β of μ is an anti fuzzy bi-ideal of R.*

Theorem 4. *The intersection of an arbitrary collection of anti fuzzy bi-ideals of a ring R is an anti fuzzy bi-ideal of R if it is not empty.*

Proof. Let $\mu_i\,(i \in I)$ be an arbitrary collection of anti fuzzy bi-ideals of R and $\mu = \underset{i\in I}{\cap}\mu_i$ be not empty. Let $x, y \in R$.

Now

$$\mu(x-y) = \left[\underset{i\in I}{\cap}\mu_i\right](x-y) = \inf_{i\in I}\{\mu_i(x-y)\} \leq \inf_{i\in I}\left[\max\{\mu_i(x), \mu_i(y)\}\right]$$

$$= \max\left[\inf_{i\in I}\{\mu_i(x)\}, \inf_{i\in I}\{\mu_i(y)\}\right] = \max\left\{\left[\underset{i\in I}{\cap}\mu_i\right](x), \left[\underset{i\in I}{\cap}\mu_i\right](y)\right\}.$$

Again

$$\mu(xy) = [\cap\mu_i](xy) = \inf_{i\in I}\{\mu_i(xy)\} \leq \inf_{i\in I}[\max\{\mu_i(x), \mu_i(y)\}]$$
$$= \max\left[\inf_{i\in I}\{\mu_i(x)\}, \inf_{i\in I}\{\mu_i(y)\}\right] = \max\left\{\left[\bigcap_{i\in I}\mu_i\right](x), \left[\bigcap_{i\in I}\mu_i\right](y)\right\}.$$

Also let $x, y, z \in R$. Then

$$\mu(xyz) = \left[\bigcap_{i\in I}\mu_i\right](xyz) = \inf_{i\in I}\{\mu_i(xyz)\} \leq \inf_{i\in I}[\max\{\mu_i(x), \mu_i(z)\}]$$
$$= \max\left[\inf_{i\in I}\{\mu_i(x)\}, \inf_{i\in I}\{\mu_i(z)\}\right] = \max\left\{\left[\bigcap_{i\in I}\mu_i\right](x), \left[\bigcap_{i\in I}\mu_i\right](z)\right\}.$$

Thus $\mu = \bigcap_{i\in I}\mu_i$ is an anti fuzzy bi-ideal of R. □

Theorem 5. *The fuzzy magnified translation of the intersection of an arbitrary collection of anti fuzzy bi-ideals of a ring R is an anti fuzzy bi-ideal of R if it is not empty.*

Proof. The proof of Theorem 5 directly follows from Theorem 2 and Theorem 4. □

Corollary 5. *The fuzzy translation of the intersection of an arbitrary collection of anti fuzzy bi-ideals of a ring R is an anti fuzzy bi-ideal of R if it is not empty.*

Corollary 6. *The fuzzy multiplication of the intersection of an arbitrary collection of anti fuzzy bi-ideals of a ring R is an anti fuzzy bi-ideal of R if it is not empty.*

The proofs of Corollary 5 and Corollary 6 are omitted.

Theorem 6. *Let μ be a non empty fuzzy sub set of a ring R. Then μ is a fuzzy bi-ideal of R if the fuzzy magnified translation $\mu^c_{\beta\alpha}$ of μ is a fuzzy bi-ideal of R.*

Proof. Let $\mu^c_{\beta\alpha}$ be a fuzzy bi-ideal of R.

Now for all $x, y, z \in R$,

$$\mu(x-y) = \frac{1}{\beta}\left[\mu^c_{\beta\alpha}(x-y) - \alpha\right] \geq \frac{1}{\beta}\left[\min\left\{\mu^c_{\beta\alpha}(x), \mu^c_{\beta\alpha}(y)\right\} - \alpha\right]$$
$$= \frac{1}{\beta}\left[\min\left\{\mu^c_{\beta\alpha}(x) - \alpha, \mu^c_{\beta\alpha}(y) - \alpha\right\}\right] = \min\left\{\frac{\mu^c_{\beta\alpha}(x) - \alpha}{\beta}, \frac{\mu^c_{\beta\alpha}(y) - \alpha}{\beta}\right\},$$
$$\mu(x-y) \geq \min\{\mu(x), \mu(y)\},$$

and

$$\mu(xy) = \frac{1}{\beta}\left[\mu^c_{\beta\alpha}(xy) - \alpha\right] \geq \frac{1}{\beta}\left[\min\left\{\mu^c_{\beta\alpha}(x), \mu^c_{\beta\alpha}(y)\right\} - \alpha\right]$$

$$= \frac{1}{\beta}\left[\min\left\{\mu^c_{\beta\alpha}(x) - \alpha, \mu^c_{\beta\alpha}(y) - \alpha\right\}\right] = \min\left\{\frac{\mu^c_{\beta\alpha}(x) - \alpha}{\beta}, \frac{\mu^c_{\beta\alpha}(y) - \alpha}{\beta}\right\},$$

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\}.$$

Next let $x, y, z \in R$. Then

$$\mu(xyz) = \frac{1}{\beta}\left[\mu^c_{\beta\alpha}(xyz) - \alpha\right] \geq \frac{1}{\beta}\left[\min\left\{\mu^c_{\beta\alpha}(x), \mu^c_{\beta\alpha}(z)\right\} - \alpha\right]$$

$$= \frac{1}{\beta}\left[\min\left\{\mu^c_{\beta\alpha}(x) - \alpha, \mu^c_{\beta\alpha}(z) - \alpha\right\}\right] = \min\left\{\frac{\mu^c_{\beta\alpha}(x) - \alpha}{\beta}, \frac{\mu^c_{\beta\alpha}(z) - \alpha}{\beta}\right\},$$

$$\mu(xyz) \geq \min\{\mu(x), \mu(z)\}.$$

Thus μ is a fuzzy bi-ideal of R. □

Remark 1. The converse of Theorem 6 is also true (cf. [2], Theorem 1).

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