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APPROXIMATE OPTIMIZATION OF CONVEX SET FUNCTIONS

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Abstract: We introduce the notion of ε-subdifferential for convex set function and discuss some of its properties. These properties are then utilized to derive ε-Pareto optimality conditions of KKT type for nondifferentiable multiobjective optimization problem with convex set functions.

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1. Introduction and Preliminaries

In various areas of applied mathematics, optimization and mathematical economics, a great deal of attention has been paid to the approximate (or ε-optimal) solutions of the problems. The concept of such solutions is considered as a perturbation or a satisfactory compromise with a prescribed error $\varepsilon > 0$. The notion of ε-optimal solution seems to be particularly useful for the class of optimization problems which otherwise have no optimal solution. Several

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research articles devoted to study and analyze the properties of ε-optimal solution have appeared in literature. Among them, contributions of Hiriart-Urruty [6], Loridan [12, 13], Strodiot et al [16], Yokoyama [18], Liu [11], and Dutta and Vetrivel [5], are notable.

On the other hand, optimization problems involving measurable functions defined on a σ-algebra of subsets of a finite atomless measure space $(X, \mathbf{A}, \mu)$ were shown to arise in diverse applications including electrical insulator design [2], optimal distribution of crops subject to the rainfall in the given area [14] and in shape optimization [15]. For this, there has been growing interest in optimization problems involving measurable set functions, modeled in general as

$$\text{(P)} \qquad \begin{aligned} &\text{Minimize} \quad F(S) \\ &\text{subject to} \quad G_i(S) \leqq 0, \quad 1 \leqq i \leqq r, \end{aligned}$$

where $F, G_i : \mathbf{A} \to \mathbf{R}$, $1 \leqq i \leqq r$.

Any set S of a measure space $(X, \mathbf{A}, \mu)$ can be identified with its characteristic function $\chi_S \in L_\infty(X, \mathbf{A}, \mu)$, therefore, by defining functionals

$$f(\chi_S) := F(S) \quad \text{and} \quad g_i(\chi_S) := G_i(S), \quad 1 \leqq i \leqq r,$$

the problem (P) can be transformed to an optimization problem over an abstract space as

$$\begin{aligned} &\text{Minimize} \quad f(x^*) \\ &\text{subject to} \quad g_i(x^*) \leqq 0, \quad 1 \leqq i \leqq r, \end{aligned}$$

where $f, g_i : L_\infty(X, \mathbf{A}, \mu) \to \mathbf{R}$, and $x^* \in \mathbf{M} = \{\chi_S : S \in \mathbf{A}\}$.

The set $\mathbf{M}$ is extremely poor for applying standard optimization theory. $\mathbf{M}$ is neither convex, nor open and not even possess linear space structure. However, $\mathbf{M}$ is a closely convex subset of $L_\infty(X, \mathbf{A}, \mu)$, i.e., $\lambda\chi_S + (1-\lambda)\chi_T \in w^* - \mathrm{cl}\,\mathbf{M}$, for any $S, T \in \mathbf{A}$ and $\lambda \in [0, 1]$, where $w^* - \mathrm{cl}\,\mathbf{M}$ is weak star closure of the set $\mathbf{M}$. For more details, refer to Definition 1.1 of [1] and Lemma 3.3 of [14].

This observation motivated Morris [14] to develop a general theory for optimizing set function. He defined the concept of convexity for set function and used it to derive the optimality conditions for scalar-valued set optimization problem. The methods and results advanced by Morris have been used and further extended by several authors. Many significant results for optimization problems with set functions are obtained in [3], [4], [7], [9], [10], [17]. In [8], we have investigated necessary and sufficient conditions characterizing ε-efficiency for vector optimization problems involving n-set functions.

In this paper, we are concerned with the ε-Pareto solutions for nondifferentiable multiobjective programming problem involving convex set functions. The paper is organized as follows.

In Section 2, we introduce the notion of ε-subdifferential for convex set function and investigate some of its properties using conjugate functional. Subsequently, in Section 3, we utilize these properties to derive the KKT type necessary and sufficient conditions for ε-Pareto optimality of multiobjective set optimization problem.

Throughout the paper, $(X, \mathbf{A}, \mu)$ is a finite atomless measure space, $L_1(X, \mathbf{A}, \mu)$ is separable, χ_Ω denotes the characteristic function of a set Ω, and $\langle f, \chi_\Omega \rangle$ represents $\int_\Omega f d\mu$.

In [14], Morris showed that given sets $\Omega, \Lambda \in \mathbf{A}$ and $\lambda \in [0,1]$, there exist L_∞-sequences $\{\chi_{\Omega_n}\}$ and $\{\chi_{\Lambda_n}\}$ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Omega \setminus \Lambda} \quad \text{and} \quad \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda \setminus \Omega},$$

imply

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_\Omega + (1-\lambda)\chi_\Lambda,$$

where w^* stand for weak* convergence.

The sequence $\{V_n\}$, where $V_n = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)$, is called Morris sequence associated with $(\lambda, \Omega, \Lambda)$.

Definition 1.1. (see [4]) A subfamily $\Im \subset \mathbf{A}$ is said to be a convex subfamily if for every $(\lambda, \Omega, \Lambda) \in [0,1] \times \Im \times \Im$, and every Morris sequence $\{V_n\}$ associated with $(\lambda, \Omega, \Lambda)$, there exists a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ in $\Im$.

Definition 1.2. (see [4]) A set function $F : \Im \to \mathbf{R}$, defined on a convex subfamily $\Im$ of $\mathbf{A}$, is said to be convex if given $(\lambda, \Omega, \Lambda) \in [0,1] \times \Im \times \Im$, and any Morris sequence $\{V_n\}$ associated with $(\lambda, \Omega, \Lambda)$, there exists a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ in $\Im$ such that

$$\limsup_{k \to \infty} F(V_{n_k}) \leqq \lambda F(\Omega) + (1-\lambda) F(\Lambda).$$

2. ε-Subdifferential

In this section, we introduce the notion of ε-subdifferential for nonsmooth convex set function and investigate some of its properties.

Let $F : \Im \to \mathbf{R}$ be a convex set function defined on a convex subfamily $\Im$ of $\mathbf{A}$. Define the conjugate functional

$F^* : (L_\infty)^* \to \mathbf{R} \cup \{+\infty\}$ of F as

$$F^*(f) = \sup_{S \in \Im}(\langle f, \chi_S \rangle - F(S)).$$

The notion of subgradient for nondifferentiable set function was introduced by Chou, Hsia and Lee [4].

The convex set function $F : \Im \to \mathbf{R}$ is called subdifferentiable at $S \in \Im$ if there exists an element $f \in (L_\infty)^*$ such that

$$F(\Omega) \geqq F(S) + \langle f, \chi_\Omega - \chi_S \rangle, \quad \forall\, \Omega \in \Im.$$

The element f is called subgradient of F at S, and the set of all subgradients of F at S is called subdifferential of F at S, denoted by $\partial F(S)$, i.e.

$$\partial F(S) = \{f \in (L_\infty)^* : F(\Omega) \geqq F(S) + \langle f, \chi_\Omega - \chi_S \rangle,\ \forall\, \Omega \in \Im\}.$$

Remark. Observe that

$$\partial F(S) = \{f \in (L_\infty)^* : F(\Omega) + F^*(f) = \langle f, \chi_\Omega \rangle, \forall\, \Omega \in \Im\}.$$

Proof. $F^*(f_1) = \sup_{S \in \Im}(\langle f_1, \chi_S \rangle - F(S)),\ f_1 \in (L_\infty)^*$

$$\Rightarrow \qquad F(S) + F^*(f_1) \geqq \langle f_1, \chi_S \rangle\,.$$

Also, $f_1 \in \partial F(S)$

$$\Rightarrow \qquad F(T) - F(S) \geqq \langle f_1, \chi_T - \chi_S \rangle, \qquad \forall\, T \in \Im\,,$$

$$\Rightarrow \qquad \langle f_1, \chi_S \rangle - F(S) \geqq \langle f_1, \chi_T \rangle - F(T), \qquad \forall\, T \in \Im\,.$$

Taking supremum over T, we obtain

$$\langle f_1, \chi_S \rangle - F(S) \geqq F^*(f_1).$$

Hence the result. □

We now introduce the notion of ε-subdifferential for convex set function, where $\varepsilon > 0$.

Definition 2.1. Let $F : \Im \to \mathbf{R}$ be a convex set function defined on a convex subfamily $\Im$ of $\mathbf{A}$. The ε-subdifferential of F at S is defined as

$$\partial_\varepsilon F(S) = \{f \in (L_\infty)^* : F(\Omega) \geqq F(S) + \langle f, \chi_\Omega - \chi_S \rangle - \varepsilon, \forall\, \Omega \in \Im\}.$$

Thus, an ε-subdifferential, $\partial_\varepsilon F(S)$, is just a 'perturbation by ε' of that of $\partial F(S)$. Moreover, $\bigcap_{\varepsilon > 0} \partial_\varepsilon F(S) = \partial F(S)$.

The following inequality can be derived from the above definition immediately.

Proposition 2.1. *For any* $S \in \Im$,

$$f \in \partial_\varepsilon F(S) \quad \Leftrightarrow \quad F(S) + F^*(f) \leqq \langle f, \chi_S \rangle + \varepsilon.$$

Proof. The result follows from the following chain of equivalences

$$\begin{aligned} f \in \partial_\varepsilon F(S) &\Leftrightarrow F(\Omega) \geqq F(S) + \langle f, \chi_\Omega - \chi_S \rangle - \varepsilon, \quad \forall\, \Omega \in \Im \\ &\Leftrightarrow \sup_{\Omega \in \Im}(\langle f, \chi_\Omega \rangle - F(\Omega)) \leqq \langle f, \chi_S \rangle - F(S) + \varepsilon \\ &\Leftrightarrow F^*(f) + F(S) \leqq \langle f, \chi_S \rangle + \varepsilon. \end{aligned}$$

□

The next result illustrate the connection between the ε-subdifferential of F and the epigraph of its conjugate functional F^*, denoted by $[F^*, (L_\infty)^*]$, defined as

$$[F^*, (L_\infty)^*] = \{(r, f) \in R \times (L_\infty)^* : F^*(f) \leqq r\}.$$

Proposition 2.2. *Let $F : \Im \to \mathbf{R}$ be a convex set function on a convex subfamily $\Im$ of $\mathbf{A}$, and let $S \in \Im$. Then,*

$$[F^*, (L_\infty)^*] = \bigcup_{\varepsilon \geqq 0} \{(\varepsilon + \langle f, \chi_S \rangle - F(S), f) : f \in \partial_\varepsilon F(S)\}.$$

Proof. Let $(r, f) \in [F^*, (L_\infty^*]$. Then $F^*(f) \leqq r$.

From the definition of conjugate function, for each $\Omega \in \Im$,

$$F^*(f) \geqq \langle f, \chi_\Omega \rangle - F(\Omega).$$

Thus, for each $\Omega \in \Im$, we have

$$\langle f, \chi_\Omega \rangle - F(\Omega) \leqq r.$$

Take $\hat{\varepsilon} = r - \langle f, \chi_S \rangle + F(S) \geqq 0$. Then for each $\Omega \in \Im$,

$$\begin{aligned} F(\Omega) &\geqq \langle f, \chi_\Omega \rangle - r \\ &= \langle f, \chi_\Omega \rangle - (\hat{\varepsilon} + \langle f, \chi_S \rangle - F(S)) \\ &= F(S) + \langle f, \chi_\Omega - \chi_S \rangle - \hat{\varepsilon} \end{aligned}$$

Consequently, $f \in \partial_{\hat{\varepsilon}} F(S)$.

We also have $F^*(f) \leqq r = \langle f, \chi_S \rangle - F(S) + \hat{\varepsilon}$. Hence, $[F^*, (L_\infty)^*] \subseteq \bigcup_{\varepsilon \geqq 0} \{(\varepsilon + \langle f, \chi_S \rangle - F(S), f) : f \in \partial_\varepsilon F(S)\}$.

Conversely, let there exists $\bar{\varepsilon} \geqq 0$ such that $f \in \partial_{\bar{\varepsilon}} F(S)$ and $r = \bar{\varepsilon} + \langle f, \chi_S \rangle - F(S)$. It follows from Proposition 2.1 that

$$F^*(f) + F(S) \leqq \langle f, \chi_S \rangle + \bar{\varepsilon},$$

implying $F^*(f) \leqq r$.

Thus, $(r, f) \in [F^*, (L_\infty)^*]$. Hence the result. □

Proposition 2.3. *Let $F, G : \Im \to \mathbf{R}$ be convex set functions on a convex* *subfamily* $\Im$ *of* $\mathbf{A}$. *Then, for any* $f \in (L_\infty)^*$,

$$(F+G)^*(f) \leqq \operatorname{Inf}\{F^*(f_1)+G^*(f_2) : f_1+f_2=f\}. \tag{2.1}$$

Proof. Let $f_1, f_2 \in (L_\infty)^*$. By definition of conjugate functional, we obtain

$$\begin{aligned} F^*(f_1) &\geqq \langle f_1, \chi_S\rangle - F(S), \qquad \forall\, S \in \Im, \\ G^*(f_2) &\geqq \langle f_2, \chi_S\rangle - G(S), \qquad \forall\, S \in \Im. \end{aligned}$$

Therefore,

$$F^*(f_1)+G^*(f_2) \geqq \langle f, \chi_S\rangle - (F+G)(S),\ \forall S \in \Im,\ f = f_1+f_2$$

giving $(F+G)^*(f) \leqq F^*(f_1)+G^*(f_2)$, $f=f_1+f_2$. Thus (2.1) holds. □

We use the following theorem due to Lai and Lin [9] as basic tool to prove the next proposition which gives the conditions under which (2.1) is satisfied as an equation.

Theorem 2.1. (see [9]) *Let* $F, G : \Im \to \mathbf{R}$ *be* w^**-continuous set functions on a convex subfamily* $\Im$ *of* $\mathbf{A}^n$. *Then*

$$\partial(F+G)(S) = \partial F(S) + \partial G(S), \qquad \forall\, S \in \Im.$$

Proposition 2.4. *Let* $F, G : \Im \to \mathbf{R}$ *be* w^**-continuous convex set functions on a convex subfamily* $\Im$ *of* $\mathbf{A}$, *and let for any* $f \in (L_\infty)^*$, *there exists* $\widetilde{S}_f \in \Im$ *such that* $f \in \partial(F+G)(\widetilde{S}_f)$. *Then equality hold in* (2.1).

Proof. Since F and G both are w^*-continuous convex set functions, hence, by Theorem 9, p. 566 of Lai and Lin [9], we have

$$\partial(F+G)(S) = \partial F(S) + \partial G(S), \qquad \forall\, S \in \Im. \tag{2.2}$$

It follows from the given condition and (2.2) that for any $f \in (L_\infty)^*$, there exists $\widetilde{S}_f \in \Im$ such that

$$f \in \partial F(\widetilde{S}_f) + \partial G(\widetilde{S}_f).$$

Thus there exists $f_1 \in \partial F(\widetilde{S}_f)$ and $f_2 \in \partial G(\widetilde{S}_f)$ such that $f = f_1 + f_2$. Moreover,

$$\begin{aligned} f_1 \in \partial F(\widetilde{S}_f) &\quad\Rightarrow\quad F(\widetilde{S}_f) + F^*(f_1) = \langle f_1, \chi_{\widetilde{S}_f}\rangle, \\ f_2 \in \partial G(\widetilde{S}_f) &\quad\Rightarrow\quad G(\widetilde{S}_f) + G^*(f_2) = \langle f_2, \chi_{\widetilde{S}_f}\rangle. \end{aligned}$$

But, we also have

$$f \in \partial(F+G)(\widetilde{S}_f) \Rightarrow (F+G)(\widetilde{S}_f) + (F+G)^*(f) = \langle f, \chi_{\widetilde{S}_f}\rangle.$$

Hence,

$$\operatorname{Inf}\{F^*(f_1)+G^*(f_2) : f_1+f_2=f\} \leqq (F+G)^*(f).$$

This completes the proof. □

The next result giving sum-rule for ε-subdifferentials of w^*-continuous convex set functions plays an important role in developing approximate optimality conditions for optimization problems in the later sections.

Theorem 2.2. *Let $F, G : \Im \to \mathbf{R}$ be w^*-continuous convex set functions on a convex subfamily $\Im$ of $\mathbf{A}$. Further, let for any $\varepsilon > 0$ and for any $f \in \partial_\varepsilon(F+G)(S)$, there exists $\widetilde{S}_f \in \Im$ such that $f \in \partial(F+G)(\widetilde{S}_f)$. Then*

$$\partial_\varepsilon(F+G)(S) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2 \geqq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon}} \{\partial_{\varepsilon_1}F(S) + \partial_{\varepsilon_2}G(S)\}.$$

Proof. Under given conditions, we have seen that

$$(F+G)^*(f) = \mathrm{Inf}\{F^*(f_1) + G^*(f_2) : f_1 + f_2 = f\}.$$

It is noteworthy from the proofs of Proposition 2.3 and Proposition 2.4 that infimum on the right hand side is attained for some $f_1, f_2 \in (L_\infty)^*$, i.e.,

$$(F+G)^*(f) = F^*(f_1) + G^*(f_2), \qquad f_1 + f_2 = f\,.$$

Now, let $f \in \partial_\varepsilon(F+G)(S)$

$$\begin{aligned}
&\Rightarrow \quad (F+G)(S) + (F+G)^*(f) \leqq \langle f, \chi_S\rangle + \varepsilon \\
&\Rightarrow \quad F(S) + G(S) + F^*(f_1) + G^*(f_2) \leqq \langle f_1, \chi_S\rangle + \langle f_2, \chi_S\rangle + \varepsilon \\
&\Rightarrow \quad (F(S) + F^*(f_1) - \langle f_1, \chi_S\rangle) + (G(S) + G^*(f_2) - \langle f_2, \chi_S\rangle) \leqq \varepsilon.
\end{aligned}$$

Since each term in the braces is non negative, we get

$$\begin{aligned}
F(S) + F^*(f_1) - \langle f_1, \chi_S\rangle &\leqq \varepsilon_1, \\
G(S) + G^*(f_2) - \langle f_2, \chi_S\rangle &\leqq \varepsilon_2, \\
\varepsilon_1 + \varepsilon_2 &= \varepsilon.
\end{aligned}$$

Thus, $f_1 \in \partial_{\varepsilon_1}F(S)$, $f_2 \in \partial_{\varepsilon_2}G(S)$ which implies

$$\partial_\varepsilon(F+G)(S) \subseteq \bigcup_{\substack{\varepsilon_1,\varepsilon_2 \geqq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon}} \{\partial_{\varepsilon_1}F(S) + \partial_{\varepsilon_2}G(S)\}.$$

Conversely, let there exist $\varepsilon_1, \varepsilon_2 \geqq 0$ such that $\varepsilon = \varepsilon_1 + \varepsilon_2$ and $f \in \partial_{\varepsilon_1}F(S) + \partial_{\varepsilon_2}G(S)$. Then, $f = \widetilde{f}_1 + \widetilde{f}_2$, for some $\widetilde{f}_1 \in \partial_{\varepsilon_1}F(S)$, $\widetilde{f}_2 \in \partial_{\varepsilon_2}G(S)$. So, we have

$$\begin{aligned}
F(S) + F^*(\widetilde{f}_1) &\leqq \langle \widetilde{f}_1, \chi_S\rangle + \varepsilon_1, \\
G(S) + G^*(\widetilde{f}_2) &\leqq \langle \widetilde{f}_2, \chi_S\rangle + \varepsilon_2.
\end{aligned}$$

Therefore, by above inequalities and Proposition 2.3, we have

$$\begin{aligned}
(F+G)(S) + (F+G)^*(f) &\leqq (F+G)(S) + F^*(\widetilde{f}_1) + G^*(\widetilde{f}_2) \\
&\leqq \langle f, \chi_S\rangle + \varepsilon
\end{aligned}$$

giving $f \in \partial_\varepsilon(F+G)(S)$.

Thus,

$$\bigcup_{\substack{\varepsilon_1,\varepsilon_2 \geqq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \{\partial_{\varepsilon_1} F(S) + \partial_{\varepsilon_2} G(S)\} \subseteq \partial_\varepsilon (F+G)(S).$$

This completes the proof. □

3. ε-Pareto Optimality Conditions

In this section, we utilize the results of Section 2 to derive ε-Pareto optimality of KKT-type for multiobjective optimization problem involving convex set functions.

Consider the multiobjective optimization problem

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & F(S) = (F_1(S), F_2(S), \ldots, F_p(S)) \\ & \text{subject to} & G_i(S) \leqq 0, \quad 1 \leqq i \leqq q, \\ & & S \in \pounds. \end{array}$$

We assume that for each k, $1 \leqq k \leqq p$, and for each i, $1 \leqq i \leqq q$, the functions $F_k, G_i : \pounds \to \mathbf{R}$ are w^*-continuous convex set functions on a convex subfamily $\pounds$ of $\mathbf{A}$. The feasible set of (MP)

$$\Im = \{S \in \pounds : G_i(S) \leqq 0,\ 1 \leqq i \leqq q\}$$

is assumed to be nonempty. Also each $F_k, 1 \leqq k \leqq p$, is bounded from below on $\Im$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p) > 0$ be the permissible error in the objective function value.

For the problem (MP), we define the ε-Pareto solution as follows:

Definition 3.1. $\bar{S} \in \Im$ is called an ε-Pareto solution of (MP) if there exist no $S \in \Im$ such that $F(S) + \varepsilon \leq F(\bar{S})$, i.e., for any $S \in \Im$, it is not possible to have

$$\begin{array}{rcll} F_k(S) + \varepsilon_k & \leqq & F_k(\bar{S}), & \forall\, k, 1 \leqq k \leqq p, k \neq j\,, \\ F_j(S) + \varepsilon_j & < & F_j(\bar{S}), & \text{for some } \ j. \end{array}$$

When $\varepsilon = 0$, the ε-Pareto solution reduces to the well known Pareto solution (or efficient solution) for (MP) (see [17]). When $p = 1$, ε-Pareto solution for (MP) becomes ε-optimal solution defined as

$$F(\bar{S}) \leqq \inf_{S \in \Im} F(S) + \varepsilon.$$

The following lemma characterizes approximate solution of (MP) via scalarization.

Lemma 3.1. *$\bar{S} \in \Im$ is an ε-Pareto solution of* (MP) *if and only if*

$$\sum_{k=1}^{p} F_k(S) + \sum_{k=1}^{p} \varepsilon_k \geqq \sum_{k=1}^{p} F_k(\bar{S}),\ \forall\ S \in \Im \cap \Im(\varepsilon, \bar{S}), \tag{3.1}$$

$$\Im(\varepsilon, \bar{S}) = \{S \in \pounds : F_k(S) \leqq F_k(\bar{S}) - \varepsilon_k, 1 \leqq k \leqq p\}.$$

Proof. Let $\bar{S}$ be an ε-Pareto solution of (MP) and let $S \in \Im \cap \Im(\varepsilon, \bar{S})$. Then

$$F_k(S) + \varepsilon_k \leqq F_k(\bar{S}).$$

But $\bar{S}$ is an ε-Pareto solution of (MP), hence,

$$F_k(S) + \varepsilon_k = F_k(\bar{S}), \quad \forall\ k,\ 1 \leqq k \leqq p$$

implying (3.1).

Conversely, let (3.1) holds. Now, if $\bar{S}$ is not an ε-Pareto solution of (MP) then there exists $\widetilde{S} \in \Im$ and an index j such that

$$F_k(\widetilde{S}) + \varepsilon_k \leqq F_k(\bar{S}), \qquad \forall\ k,\ 1 \leqq k \leqq p,\ k \neq j\,,$$

$$F_j(\widetilde{S}) + \varepsilon_j < F_j(\bar{S}).$$

Clearly, $\widetilde{S} \in \Im \cap \Im(\varepsilon, \bar{S})$, and we have

$$\sum_{k=1}^{p} F_k(\widetilde{S}) + \sum_{k=1}^{p} \varepsilon_k < \sum_{k=1}^{p} F_k(\bar{S})$$

which contradicts (3.1). Hence the result. □

Theorem 3.1. *Let $\bar{S} \in \Im$ be an ε-Pareto solution of (MP) and let equality holds in (2.1). Then there exist $(\lambda_1, \lambda_2, \ldots, \lambda_p, \mu_1, \mu_2, \ldots, \mu_q) \neq 0$ and $\bar{\varepsilon}_k \geqq 0$, $1 \leqq k \leqq p$, $\widetilde{\varepsilon}_i \geqq 0$, $1 \leqq i \leqq q$, such that*

$$0 \in \sum_{k=1}^{p} \partial_{\bar{\varepsilon}_k}(\lambda_k F_k)(\bar{S}) + \sum_{i=1}^{q} \partial_{\widetilde{\varepsilon}_i}(\mu_i G_i)(\bar{S}),$$

$$\sum_{k=1}^{p} \bar{\varepsilon}_k + \sum_{i=1}^{q} \widetilde{\varepsilon}_i = \sum_{k=1}^{p} \lambda_k \varepsilon_k,$$

$$-\sum_{k=1}^{p} \lambda_k \varepsilon_k \leqq \sum_{i=1}^{q} \mu_i G_i(\bar{S}) \leqq 0, \quad \lambda_k \geqq 0, \quad 1 \leqq k \leqq p, \quad \mu_i \geqq 0, \quad 1 \leqq i \leqq q.$$

Proof. Define sets

$$A_\varepsilon \;=\; \{v = (v_0, v_1, \ldots, v_p, w_1, \ldots, w_q) \in \mathbf{R}^{1+p+q} : \exists\ S \in \Im \text{ such that}$$

$$v_0 > \sum_{k=1}^{p} F_k(S) + \sum_{k=1}^{p} \varepsilon_k,\ v_k \geqq F_k(S) + \varepsilon_k,\ 1 \leqq k \leqq p,$$

$$\begin{aligned} & w_i \geqq G_i(S),\ 1 \leqq i \leqq p\}\,, \\ B &= \{v=(v_0,v_1,\dots,v_p,w_1,\dots,w_q)\in \mathbf{R}^{1+p+q} : v_0 < \sum_{k=1}^{p} F_k(\bar{S}), \\ & v_k < F_k(\bar{S}),\ 1\leqq k \leqq p,\ w_i<0,\ 1\leqq i\leqq q\}. \end{aligned}$$

We will show that $\bar{A}_\varepsilon$ in $\mathbf{R}^{1+p+q}$ is a convex set.

Choose $\delta > 0$, $\lambda \in [0,1]$ and $u, \hat{u} \in \bar{A}_\varepsilon$. Then there exist $S_1, S_2 \in \Im$ with

$$\begin{aligned} u &\geqq \left(\sum_{k=1}^{p} F_k(S_1) + \sum_{k=1}^{p} \varepsilon_k - \frac{\delta}{2},\ F_1(S_1)+\varepsilon_1-\frac{\delta}{2},\dots,\right. \\ &\qquad \left. F_p(S_1)+\varepsilon_p-\frac{\delta}{2},\ G_1(S_1)-\frac{\delta}{2},\dots,G_q(S_1)-\frac{\delta}{2}\right)\,, \\ \hat{u} &\geqq \left(\sum_{k=1}^{p} F_k(S_2) + \sum_{k=1}^{p} \varepsilon_k - \frac{\delta}{2},\ F_1(S_2)+\varepsilon_1-\frac{\delta}{2},\dots,\right. \\ &\qquad \left. F_p(S_2)+\varepsilon_p-\frac{\delta}{2},\ G_1(S_2)-\frac{\delta}{2},\dots,G_q(S_2)-\frac{\delta}{2}\right)\,. \end{aligned}$$

Since $\Im$ is a convex subfamily and each F_k, G_i are convex set functions on $\Im$, so there exists Morris sequence $\{V^n\}$ associated with (λ, S_1, S_2) such that

$$\chi_{S_1^n} \xrightarrow{w^*} \lambda\chi_{S_1\setminus S_2}, \qquad \chi_{S_2^n} \xrightarrow{w^*} (1-\lambda)\chi_{S_2\setminus S_1}.$$

Moreover, for the Morris sequence $V^n = S_1^n \cup S_2^n \cup (S_1\cap S_2)$, we have $\chi_{V^n} \xrightarrow{w^*} \lambda\chi_{S_1} + (1-\lambda)\chi_{S_2}$, and the functions satisfy the following inequalities

$$\begin{aligned} \limsup_{r\to\infty} F_k(V^{n_r}) &\leqq \lambda F_k(S_1) + (1-\lambda)F_k(S_2), \quad 1\leqq k\leqq p, \\ \limsup_{r\to\infty} G_i(V^{n_r}) &\leqq \lambda G_i(S_1) + (1-\lambda)G_i(S_2), \quad 1\leqq i\leqq q, \\ \limsup_{r\to\infty} \sum_{k=1}^{p} F_k(V^{n_r}) &\leqq \lambda \sum_{k=1}^{p} F_k(S_1) + (1-\lambda)\sum_{k=1}^{p} F_k(S_2)\,. \end{aligned}$$

So, for sufficiently large n_r, we have

$$\begin{aligned} F_k(V^{n_r}) &\leqq \lambda F_k(S_1) + (1-\lambda)F_k(S_2) + \frac{\delta}{2}, \quad 1\leqq k\leqq p\,, \\ G_i(V^{n_r}) &\leqq \lambda G_i(S_1) + (1-\lambda)G_i(S_2) + \frac{\delta}{2}, \quad 1\leqq i\leqq q\,, \\ \sum_{k=1}^{p} F_k(V^{n_r}) &\leqq \lambda \sum_{k=1}^{p} F_k(S_1) + (1-\lambda)\sum_{k=1}^{p} F_k(S_2) + \frac{\delta}{2}. \end{aligned}$$

Thus, there exists $V^{n_r} \in \Im$ such that

$$\lambda u + (1-\lambda)\hat{u} \geqq \left(\sum_{k=1}^{p} F_k(V^{n_r}) + \sum_{k=1}^{p} \varepsilon_k - \delta, F_1(V^{n_r}) + \varepsilon_1 - \delta, \dots, \right.$$
$$\left. F_p(V^{n_r}) + \varepsilon_p - \delta, G_1(V^{n_r}) - \delta, \dots, G_q(V^{n_r}) - \delta \right).$$

But $\delta > 0$ is arbitrary, hence,

$$\lambda u + (1-\lambda)\hat{u} \in \bar{A}_\varepsilon.$$

Clearly, B is a convex set. Further, in view of Lemma 3.1,

$$\bar{A}_\varepsilon \cap B = \phi\,.$$

By the standard separation theorem, there exists a vector $(\beta_0, \beta, \mu) \in \mathbf{R}^{1+p+q}$ $(\beta_0, \beta, \mu) \geqq 0$, $(\beta_0, \beta, \mu) \neq 0$ such that

$$\langle (\beta_0, \beta, \mu), v \rangle \geqq \langle (\beta_0, \beta, \mu), v' \rangle, \quad \forall\, v \in A_\varepsilon,\ v' \in B\,,$$

$$v = \left(\sum_{k=1}^{p} F_k(S) + \sum_{k=1}^{p} \varepsilon_k, F_1(S) + \varepsilon_1, \dots, F_p(S) + \varepsilon_p, G_1(S), \dots, G_q(S) \right),$$

$$v' = \left(\sum_{k=1}^{p} F_k(\bar{S}) - \eta, F_1(\bar{S}) - \eta, \dots, F_p(\bar{S}) - \eta, -\eta, \dots, -\eta \right),\ \eta > 0.$$

Then,

$$\beta_0 \left(\sum_{k=1}^{p} F_k(S) + \sum_{k=1}^{p} \varepsilon_k \right) + \sum_{k=1}^{p} \beta_k (F_k(S) + \varepsilon_k) + \sum_{i=1}^{q} \mu_i G_i(S)$$
$$\geqq \beta_0 \left(\sum_{k=1}^{p} F_k(\bar{S}) - \eta \right) + \left(\sum_{k=1}^{p} \beta_k F_k(\bar{S}) - \eta \right) - \sum_{i=1}^{q} \mu_i \eta$$
$$= \beta_0 \sum_{k=1}^{p} F_k(\bar{S}) + \sum_{k=1}^{p} \beta_k F_k(\bar{S}) - \eta \left(\beta_0 + \sum_{k=1}^{p} \beta_k + \sum_{i=1}^{q} \mu_i \right).$$

Since $\eta > 0$ is arbitrary, hence

$$\sum_{k=1}^{p} (\beta_0 + \beta_k)(F_k(S) - F_k(\bar{S}) + \varepsilon_k) + \sum_{i=1}^{q} \mu_i G_i(S) \geqq 0, \quad \forall\ S \in \Im.$$

Set $\lambda_k = \beta_0 + \beta_k \geqq 0$, we get

$$\sum_{k=1}^{p} \lambda_k (F_k(S) - F_k(\bar{S}) + \varepsilon_k) + \sum_{i=1}^{q} \mu_i G_i(S) \geqq 0, \quad \forall\ S \in \Im. \tag{3.2}$$

Let $S = \bar{S}$ in the above relation, we get

$$\sum_{i=1}^{q} \mu_i G_i(\bar{S}) \geqq -\sum_{k=1}^{p} \lambda_k \varepsilon_k \,.$$

This along with the facts that $G_i(\bar{S}) \leqq 0$, $\mu_i \geqq 0$, $1 \leqq i \leqq q$, yields

$$-\sum_{k=1}^{p} \lambda_k \varepsilon_k \leqq \sum_{i=1}^{q} \mu_i G_i(\bar{S}) \leqq 0\,. \tag{3.3}$$

Also, from (3.2),

$$\sum_{k=1}^{p} \lambda_k F_k(S) + \sum_{i=1}^{q} \mu_i G_i(S) + \varepsilon' \geqq \sum_{k=1}^{p} \lambda_k F_k(\bar{S}), \quad \varepsilon' = \sum_{k=1}^{p} \lambda_k \varepsilon_k$$

$$\Rightarrow \quad \sum_{k=1}^{p} \lambda_k F_k(S) + \sum_{i=1}^{q} \mu_i G_i(S) + \varepsilon' \geqq \sum_{k=1}^{p} \lambda_k F_k(\bar{S}) + \sum_{i=1}^{q} \mu_i G_i(\bar{S}), \forall\ S \in \Im.$$

Thus $\bar{S}$ is an ε'-optimal solution of the problem

$$\operatorname*{Minimize}_{S \in \Im} \left(\sum_{k=1}^{p} \lambda_k F_k + \sum_{i=1}^{q} \mu_i G_i \right)(S).$$

Consequently,

$$0 \in \partial_{\varepsilon'} \left(\sum_{k=1}^{p} \lambda_k F_k + \sum_{i=1}^{q} \mu_i G_i \right)(\bar{S}).$$

By Theorem 2.2, there exists $\bar{\varepsilon}_k \geqq 0$, $1 \leqq k \leqq p$, $\widetilde{\varepsilon}_i \geqq 0$, $1 \leqq i \leqq q$, such that

$$0 \in \sum_{k=1}^{p} \partial_{\bar{\varepsilon}_k}(\lambda_k F_k)(\bar{S}) + \sum_{i=1}^{q} \partial_{\widetilde{\varepsilon}_i}(\mu_i G_i)(\bar{S})\,, \tag{3.4}$$

$$\sum_{k=1}^{p} \bar{\varepsilon}_k + \sum_{i=1}^{q} \widetilde{\varepsilon}_i = \varepsilon' = \sum_{k=1}^{p} \lambda_k \varepsilon_k\,. \tag{3.5}$$

(3.3), (3.4), (3.5) give the required result. □

Corollary 3.1. *Let the conditions of Theorem 3.1 be satisfied. Further, let $G_1, \dots, G_q$ satisfy Karlin's constraint qualification, i.e., for each $\gamma = (\gamma_1, \dots, \gamma_q) \in \mathbf{R}^q$, $\gamma \geqq 0$, $\gamma \neq 0$, there exists $\hat{S} \in \Im$ such that $\sum_{i=1}^{q} \gamma_i G_i(\hat{S}) < 0$. Then $\sum_{k=1}^{p} \lambda_k = 1$.*

Proof. If in Theorem 3.1, all $\lambda_k = 0$, $1 \leqq k \leqq p$, then from (3.2) it follows that $\sum_{i=1}^{q} \mu_i G_i(S) \geqq 0$, $\forall\ S \in \Im$, with $\mu = (\mu_1, \dots, u_q) \neq 0$, $\mu \geqq 0$.

But this contradicts the Karlin's constraint qualification assumption. Thus, $\lambda_k \neq 0$ for at least one index k, $1 \leqq k \leqq q$. Therefore, without loss of generality, we can take $\sum\limits_{k=1}^{p} \lambda_k = 1$. □

Next we prove the sufficiency of the KKT-necessary optimality conditions developed in Theorem 3.1. However, the sufficient optimality conditions will provide only the weaker solution of the problem (MP).

Theorem 3.2. *Assume that the following conditions hold*

$$0 \in \sum_{k=1}^{p} \partial_{\bar{\varepsilon}_k}(\lambda_k F_k)(\bar{S}) + \sum_{i=1}^{q} \partial_{\widetilde{\varepsilon}_i}(\mu_i G_i)(\bar{S}), \qquad \sum_{k=1}^{p} \bar{\varepsilon}_k + \sum_{i=1}^{q} \widetilde{\varepsilon}_i = \sum_{k=1}^{p} \lambda_k \varepsilon_k,$$

$$-\sum_{k=1}^{p} \lambda_k \varepsilon_k \leqq \sum_{i=1}^{q} \mu_i G_i(\bar{S}) \leqq 0, \qquad \sum_{k=1}^{p} \lambda_k = 1, \lambda_k \geqq 0, 1 \leqq k \leqq p,$$

$$\mu_i \geqq 0, \quad 1 \leqq i \leqq q,$$

$$\bar{\varepsilon}_k \geqq 0, \quad 1 \leqq k \leqq p, \quad \widetilde{\varepsilon}_i \geqq 0, \quad 1 \leqq i \leqq q, \quad \bar{S} \in \mathbf{A}.$$

Then $\bar{S}$ is 2ε-weak Pareto solution of (MP).

Proof. It follows from the given condition that there exist $\varphi_k \in \partial_{\bar{\varepsilon}_k}(\lambda_k F_k)(\bar{S})$ and $\psi_i \in \partial_{\widetilde{\varepsilon}_i}(\mu_i G_i)(\bar{S})$ such that

$$0 = \sum_{k=1}^{p} \varphi_k + \sum_{i=1}^{q} \psi_i \,.$$

Also, for each $S \in \mathbf{A}$,

$$\sum_{k=1}^{p} \lambda_k F_k(S) - \sum_{k=1}^{p} \lambda_k F_k(\bar{S}) \geqq \sum_{k=1}^{p} \langle \varphi_k, \chi_S - \chi_{\bar{S}} \rangle - \sum_{k=1}^{p} \bar{\varepsilon}_k,$$

$$\sum_{k=1}^{q} \mu_i G_i(S) - \sum_{i=1}^{q} \mu_i G_i(\bar{S}) \geqq \sum_{k=1}^{q} \langle \Psi_i, \chi_S - \chi_{\bar{S}} \rangle - \sum_{k=1}^{p} \widetilde{\varepsilon}_i.$$

Adding the above inequalities, and using the facts that $S \in \mathbf{A}$, $\mu_i \geqq 0$, $1 \leqq i \leqq q$, we get

$$\sum_{k=1}^{p} \lambda_k F_k(\bar{S}) + \sum_{i=1}^{q} \mu_i G_i(\bar{S}) \leqq \sum_{k=1}^{p} \lambda_k F_k(S) + \sum_{k=1}^{p} \lambda_k \varepsilon_k, \quad \forall\ S \in \mathbf{A}\,,$$

which implies

$$\sum_{k=1}^{p} \lambda_k F_k(\bar{S}) \leqq \sum_{k=1}^{p} \lambda_k F_k(S) + 2 \sum_{k=1}^{p} \lambda_k \varepsilon_k, \quad \forall\ S \in \mathbf{A}\,. \tag{3.6}$$

Now, if $\bar{S}$ is not 2ε-weak Pareto solution of (MP), then there would exists $S' \in A$ such that

$$F_k(S') + 2\varepsilon_k < F_k(\bar{S}), \quad \forall\ k, 1 \leqq k \leqq p\,.$$

Multiplying by $\lambda_k \geqq 0$, $\sum\limits_{k=1}^{p} \lambda_k = 1$, and adding we get

$$\sum_{k=1}^{p} \lambda_k F_k(\bar{S}) > \sum_{k=1}^{p} \lambda_k F_k(S') + 2\sum_{k=1}^{p} \lambda_k \varepsilon_k$$

which contradicts (3.6). Hence the result. □

4. Concluding Remarks

As a special case of multiobjective set optimization problem (MP), consider an unconstrained scalar-valued set optimal problem of the form

$$\text{(SP)} \quad \text{Minimize} \quad F(S) = F(S) = u\left(\int_S v_1 d\mu, \ldots, \int_S v_n d\mu\right)\,,$$

where $u : \mathbf{R}^n \to \mathbf{R}$ is differentiable and $v_i \in L_\infty(X, \mathbf{A}, \mu)$, $1 \leqq i \leqq n$. Since the set

$$\mathbf{E} = \text{Range}\left\{\left[\int_S v_1 d\mu, \ldots, \int_S v_n d\mu\right] : S \in \mathbf{A}\right\}$$

is a convex compact subset of $\mathbf{R^n}$, the problem (SP) reduces to finding an optimal solution of a nonlinear programming problem over a convex set $\mathbf{E}$.

A numerical technique to solve the problem (SP) was developed by Morris [14]. It was shown that the optimal solution set of (SP) can be obtained via solution of a fixed point problem in $\mathbf{R^n}$. To apply this technique, Morris suggested to use suitable perturbation in the functions u, $v_1, \ldots, v_n$ without significantly distorting the original problem. However, the solution set thus obtained for perturbation differ eligibly from the solutions of the original problem.

This observation motivated us to introduce the notion of "approximate solution" for set optimization problem in this paper.

We have defined the new concepts of ε-subdifferential for convex set functions and ε-Pareto optimality for multiobjective optimization problem with set functions. Properties of ε-subdifferential are investigated and are then applied to characterize ε-Pareto optimality of convex multiobjective programming problem.

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