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ON THE L^*-ORDER OF MEROMORPHIC FUNCTIONS

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Abstract: In this paper we establish the relationship between the L^*-order (L^*-type) of two meromorphic functions based on their sharing of values.

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1. Introduction, Definitions and Notations

Let f and g be two non constant meromorphic functions defined on the open complex plane $\mathbb{C}$. If for $a \in \mathbb{C} \cup \{\infty\}$ f and g have the same a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities). We do not explain the standard definitions and notations of the value distribution theory as those are available in Hayman [1]. To start our paper we just recall the following definitions:

Definition 1. The order ρ_f and lower order λ_f of a meromorphic function f are defined as

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$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna's characteristic function of f.

Definition 2. The type σ_f of a meromorphic function f is defined as follows:

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

Definition 3. The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of a meromorphic function f are defined in the following way:

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r},$$

where

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and} \\ \log^{[0]} x &= x. \end{aligned}$$

Somasundaram and Thamizharasi [2] introduced the notions of L-order for meromorphic functions where $L = L(r)$ is a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Their definitions are as follows:

Definition 4. (see Somasundaram and Thamizharasi [2]) The L-order ρ_f^L and L-lower order λ_f^L of a meromorphic function f are defined as

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log [rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log [rL(r)]}.$$

Similarly one can define the L-hyper order $\bar{\rho}_f^L$ and L-hyper lower order $\bar{\lambda}_f^L$ of a meromorphic function f in the following way:

$$\bar{\rho}_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log [rL(r)]} \quad \text{and} \quad \bar{\lambda}_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log [rL(r)]}.$$

Definition 5. (see Somasundaram and Thamizharasi [2]) The L-type σ_f^L of a meromorphic function f is defined as follows

$$\sigma_f^L = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f^L}}, \quad \text{where } 0 < \rho_f^L < \infty.$$

The more generalised concept of L-order and L-lower order of a meromorphic function f are L^*-order and L^*-lower order and their definitions are as follows:

Definition 6. The L^*-order $\rho_f^{L^*}$ and L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}.$$

Analogously the definitions of L^*-hyper order $\bar{\rho}_f^{L^*}$ and L^*-hyper lower order $\bar{\lambda}_f^{L^*}$ of a meromorphic function f may be given.

Definition 7. The L^*-type $\sigma_f^{L^*}$ of a meromorphic function f is defined in the following way

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad \text{where } 0 < \rho_f^{L^*} < \infty.$$

In this paper we wish to develop the relationship between the L^*-order (L^*-type) of two meromorphic functions f and g sharing $0, 1, \infty$ CM.

2. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f and g be two non constant meromorphic functions sharing $0, 1, \infty$ CM. Then:*

(i) $\rho_f = \rho_g$, $\lambda_f = \lambda_g$, $\frac{1}{3} \le \frac{\sigma_f}{\sigma_g} \le 3$,

(ii) $\rho_f^L = \rho_g^L$, $\lambda_f^L = \lambda_g^L$, $\frac{1}{3} \le \frac{\sigma_f^L}{\sigma_g^L} \le 3$, *and*

(iii) $\rho_f^{L^*} = \rho_g^{L^*}$, $\lambda_f^{L^*} = \lambda_g^{L^*}$, $\frac{1}{3} \le \frac{\sigma_f^{L^*}}{\sigma_g^{L^*}} \le 3$,

where $0 < \rho_f < \infty$, $0 < \rho_g < \infty$, $0 < \rho_f^L < \infty$, $0 < \rho_g^L < \infty$, $0 < \rho_f^{L^*} < \infty$ *and* $0 < \rho_g^{L^*} < \infty$.

Proof. Since f and g share $0, 1, \infty$ CM, we get in view of Nevanlinna's Second Fundamental Theorem

$$\begin{aligned} T(r,f) &\le N(r,0;f) + N(r,1;f) + N(r,\infty;f) + S(r,f) \\ &= N(r,0;g) + N(r,1;g) + N(r,\infty;g) + S(r,f) \\ &\le 3T(r,g) + S(r,f) \end{aligned} \tag{1}$$

and

$$\begin{aligned} T(r,g) &\le N(r,0;g) + N(r,1;g) + N(r,\infty;g) + S(r,g) \\ &= N(r,0;f) + N(r,1;f) + N(r,\infty;f) + S(r,g) \end{aligned}$$

$$\leq \quad 3T(r,f) + S(r,g)\,. \tag{2}$$

Now from (1) and (2) we obtain that

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r} \leq \limsup_{r\to\infty}\frac{\log T(r,g)}{\log r}$$

and

$$\limsup_{r\to\infty}\frac{\log T(r,g)}{\log r} \leq \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}\,,$$

$$\rho_f \leq \rho_g \quad \text{and} \quad \rho_g \leq \rho_f,$$

from which it follows that

$$\rho_f = \rho_g.$$

Similarly we can prove that

$$\lambda_f = \lambda_g.$$

As $\rho_f = \rho_g$ in view of (1) and (2) we get that

$$\limsup_{r\to\infty}\frac{T(r,f)}{r^{\rho_f}} \leq 3\limsup_{r\to\infty}\frac{T(r,g)}{r^{\rho_g}}$$

and

$$\limsup_{r\to\infty}\frac{T(r,g)}{r^{\rho_g}} \leq 3\limsup_{r\to\infty}\frac{T(r,f)}{r^{\rho_f}},$$

where $0 < \rho_f < \infty$ and $0 < \rho_g < \infty$,

$$\sigma_f \leq 3\sigma_g \quad \text{and} \quad \sigma_g \leq 3\sigma_f\,,$$

$$\frac{1}{3} \leq \frac{\sigma_f}{\sigma_g} \leq 3.$$

This proves the first part of the theorem.

Again dividing both sides of (1) and (2) respectively by $\log[rL(r)]$ and taking limit superior we get that

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\log[rL(r)]} \leq \limsup_{r\to\infty}\frac{\log T(r,g)}{\log[rL(r)]}$$

and

$$\limsup_{r\to\infty}\frac{\log T(r,g)}{\log[rL(r)]} \leq \limsup_{r\to\infty}\frac{\log T(r,f)}{\log[rL(r)]}\,,$$

$$\rho_f^L \leq \rho_g^L \quad \text{and} \quad \rho_g^L \leq \rho_f^L,$$

from which it follows that

$$\rho_f^L = \rho_g^L.$$

Similarly it can be shown that

$$\lambda_f^L = \lambda_g^L.$$

Since $\rho_f^L = \rho_g^L$ in view of (1) and (2) we obtain that

$$\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f^L}} \leq 3 \limsup_{r\to\infty} \frac{T(r,g)}{r^{\rho_g^L}}$$

and

$$\limsup_{r\to\infty} \frac{T(r,g)}{r^{\rho_g^L}} \leq 3 \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f^L}},$$

where $0 < \rho_f^L < \infty$ and $0 < \rho_g^L < \infty$,

$$\sigma_f^L \leq 3\sigma_g^L \quad \text{and} \quad \sigma_g^L \leq 3\sigma_f^L\,,$$

$$\frac{1}{3} \leq \frac{\sigma_f^L}{\sigma_g^L} \leq 3.$$

This proves the second part of the theorem.

On dividing both sides of (1) and (2) respectively by $\log\left[re^{L(r)}\right]$ and taking limit superior we obtain that

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \leq \limsup_{r\to\infty} \frac{\log T(r,g)}{\log\left[re^{L(r)}\right]}$$

and

$$\limsup_{r\to\infty} \frac{\log T(r,g)}{\log\left[re^{L(r)}\right]} \leq \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

$$\text{i.e., } \rho_f^{L^*} \leq \rho_g^{L^*} \quad \text{and} \quad \rho_g^{L^*} \leq \rho_f^{L^*},$$

from which it follows that

$$\rho_f^{L^*} = \rho_g^{L^*}.$$

Similarly it can be shown that

$$\lambda_f^{L^*} = \lambda_g^{L^*}.$$

As $\rho_f^{L^*} = \rho_g^{L^*}$ in view of (1) and (2) we get that

$$\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f^{L^*}}} \leq 3 \limsup_{r\to\infty} \frac{T(r,g)}{r^{\rho_g^{L^*}}}$$

and

$$\limsup_{r\to\infty} \frac{T(r,g)}{r^{\rho_g^{L^*}}} \leq 3 \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f^{L^*}}},$$

where $0 < \rho_f^{L^*} < \infty$ and $0 < \rho_g^{L^*} < \infty$.

$$\text{i.e., } \sigma_f^{L^*} \leq 3\sigma_g^{L^*} \text{and } \sigma_g^{L^*} \leq 3\sigma_f^{L^*}$$

$$\text{i.e.,} \frac{1}{3} \leq \frac{\sigma_f^{L^*}}{\sigma_g^{L^*}} \leq 3.$$

This proves the third part of the theorem. □

In the line of Theorem 1 we may state the following theorem without proof.

Theorem 2. *Let f and g be two non constant meromorphic functions sharing $0, 1, \infty$ CM. Then:*

(i) $\bar{\rho}_f = \bar{\rho}_g$, $\bar{\lambda}_f = \bar{\lambda}_g$;

(ii) $\bar{\rho}_f^L = \bar{\rho}_g^L$, $\bar{\lambda}_f^L = \bar{\lambda}_g^L$ *and*

(iii) $\bar{\rho}_f^{L^*} = \bar{\rho}_g^{L^*}$, $\bar{\lambda}_f^{L^*} = \bar{\lambda}_g^{L^*}$.

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