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REDUCIBLE BIELLIPTIC CURVES

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Abstract: Here we consider reducible bielliptic curves $u : X = X_1 \cup X_2 \to C$ with C integral and $p_a(C) = 1$. For certain bidegrees (d_1, d_2) we prove the existence or non-existence of a finite morphism $f : X \to \mathbb{P}^1$ with bidegree (d_1, d_2) such that the induced morphism $(u, f) : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image.

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1. Introduction

Fix an integer $g \geq 5$. Let C be a smooth elliptic curve. Fix $S \subset C$ such that $\sharp(S) = g - 1$ and an ordering σ_i, say $P_1, \dots, P_{g-1}$, of S. Let (X_i, S_i, σ_i), $i = 1, 2$, be two copies of the triple (C, S, σ). Let X be the nodal curve obtained gluing each point of $S_1 \subset X_1$ to the corresponding point of $S_2 \subset X_2$. X is a stable curve of genus g equipped with a degree 2 morphism $u : X \to C$ such that each $u|X_i : X_i \to C$ is an isomorphism sending S_i isomorphically onto S. We say that X is a *reducible bielliptic curve of type I.* Let D be an integral projective curve such that $p_a(D) = 1$ and the unique singular point P of D is an ordinary node. For the case in which P is an ordinary cusp, see Remark 2. Fix $S \subset D_{reg}$ such that $\sharp(S) = g - 1$ and an ordering σ of S. Let (X_i, S_i, σ_i),

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$i = 1, 2$, be two copies of the triple (D, S, σ). Let X be the nodal curve obtained gluing each point of $S_1 \subset X_1$ to the corresponding point of $S_2 \subset X_2$. X is a stable curve of genus g equipped with a degree 2 morphism $u : X \to D$ such that each $u|X_i : X_i \to D$ is an isomorphism sending S_i isomorphically onto S. We say that X is a *reducible bielliptic curve of type II*. Let E be a reducible, connected and nodal curve such that $p_a(E) = 1$. Hence E has two irreducible components, say E_1 and E_2, such that $E_1 \cong E_2 \cong \mathbb{P}^1$ and $\sharp(E_1 \cap E_2) = 2$. Fix nodal projective curves Y_1, Y_2 and degree 2 finite morphisms $u_i : Y_i \to E_i$, $i = 1, 2$. We do not assume the connectedness of Y_1 or Y_2. Let Y be the nodal curve obtained gluing some of the points of $u_1^{-1}(E_1 \cap E_2) \subset Y_1$ with some of the points of $u_2^{-1}(E_1 \cap E_2) \subset Y_2$. If Y is connected, then Y is called a *bielliptic curve of type III*. Y is said to be *nice* if u_1 and u_2 are unramified at all points mapped over $E_1 \cap E_2$ and each point of $u_1^{-1}(E_1 \cap E_2)$ is glue to a unique point of $u_2^{-1}(E_1 \cap E_2)$. Fix integer $g \geq 2q \geq 4$, an integral projective curve Y and $S \subset Y_{reg}$ such that $\sharp(S) = g + 1 - 2q$. Fix an ordering, say $P_1, \ldots, P_{g+1-2q}$ of S. Fix two copies (Y_i, S_i), $i = 1, 2$, of the pair (Y, S) and let X be the reducible curve obtained gluing together each point of $S_1 \subset Y_1$ with the corresponding point of $S_2 \subset Y_2$ to get an ordinary node. Hence $p_a(X) = g$, X is connected and it has two irreducible components, X_1 and X_2. X is equipped with a morphism $u : X \to Y$ such that $u|X_i : X_i \to Y$ is an isomorphism. The triple (X, Y, u) is called a *reducible q-hyperelliptic curve with Y as its target*. Here we prove the following results.

Theorem 1. *Fix positive integers d_1, d_2 and a smooth elliptic curve C.*

(a) Assume either $d_1 = 1$ or $d_2 = 1$ or $d_1 + d_2 \leq g - 2$. Then for every reducible bielliptic curve $u : X = X_1 \cup X_2 \to C$ of type I there is no degree d finite morphism $f : X \to \mathbb{P}^1$ such that the morphism $(u, f) : X \to C \times \mathbb{P}^1$ is birational onto its image.

(b) Assume $d_1 \geq 2$, $d_2 \geq 2$ and $d_1 + d_2 \geq g - 1$. There exist a reducible bielliptic curve $u : X \to C$ of type I and a degree finite morphism $f : X \to \mathbb{P}^1$ such that $\deg(u|X_i) = d_i$, $i = 1, 2$, and the morphism $(u, f) : X \to C \times \mathbb{P}^1$ is birational onto its image.

Theorem 2. *Fix positive integers d_1, d_2 and an integral curve D such that $p_a(D) = 1$ and D has an ordinary node.*

(a) Assume $d_1 + d_2 \leq g - 2$. Then for every reducible bielliptic curve $u : X = X_1 \cup X_2 \to D$ of type II there is no finite morphism $f : X \to \mathbb{P}^1$ such that $\deg(f|X_i) = d_i$, $i = 1, 2$, and the morphism $(u, f) : X \to D \times \mathbb{P}^1$ is birational onto its image.

(b) Assume $d_1 \geq 2$, $d_2 \geq 2$ and $d_1 + d_2 \geq g - 1$. There exist a reducible bielliptic curve $u : X \to D$ of type II and a finite morphism $f : X \to \mathbb{P}^1$ such that the morphism $(u, f) : X \to D \times \mathbb{P}^1$ is birational onto its image and $\deg(f|X_i) = d_i$, $i = 1, 2$.

Theorem 3. *Let Y be an integral projective curve such that $q := p_a(Y) \geq 2$. Fix integers $d_1 \geq 3$ and $d_2 \geq 3$ such that there are very ample $M_i \in \mathrm{Pic}^{d_i}(Y)$, $i = 1, 2$. Assume $d_1 + d_2 \geq 2q - 1$ and fix an integer g such that $d_1 + d_2 \geq g + 1 - 2q$. Then there exist a q-hyperelliptic reducible curve $(X = X_1 \cup X_2, Y, u)$ with Y as its target and a finite morphism $f : X \to \mathbb{P}^1$ such that $p_a(X) = g$ and* $\deg(f|X_i) = d_i$, $i = 1, 2$.

We do not have general results concerning reducible bielliptic curves of type III.

2. The Proofs

Remark 1. Set $W := C \times \mathbb{P}^1$. Let $p_1 : W \to C$ and $p_2 : W \to \mathbb{P}^1$ denote the projections. For any $M \in \mathrm{Pic}(C)$ and any integer t set $(M, t) := p_1^*(M) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(t))$. Every line bundle on W is isomorphic to a line bundle (M, t) for some uniquely determined M, t. Fix $M \in \mathrm{Pic}^a(C)$. If $N \in \mathrm{Pic}^b(C)$, then $(M, t) \cdot (N, t') = at' + bt$. In the next few sentences we apply Künneth formula. If either $a < 0$ or $t < 0$, then $h^0(W, (M, t)) = 0$. If $a > 0$ and $t \geq 0$, then $h^1(W, (M, t)) = 0$ and $h^0(W, (M, t)) = a(t + 1)$. If $a \geq 2$ and $t \geq 0$, then (M, t) is spanned. If $a \geq 3$ and $t > 0$, then (M, t) is very ample. Assume $a \geq 2$; every integral curve in $|(M, 1)|$ is mapped isomorphically by p_1 onto C; any non-integral curve in $|(M, 1)|$ is the union of a curve mapped isomorphically by p_1 onto C and some fibers of p_1; hence a dimensional computation shows that a general element of $|(M, 1)|$ is integral.

(i) Fix integers $d_i \geq 2$ and $M_i \in \mathrm{Pic}^{d_i}(C)$, $i = 1, 2$. We saw that $|(M_i, 1)| \neq \emptyset$ and that a general member of it is isomorphic to C. We have $(M_1, 1) \cdot (M_2, 1) = d_1 + d_2$. Take a general pair $(T_1, T_2) \in |(M_1, 1)| \times |(M_2, 1)|$ such that T_1 intersects transversally T_2, i.e. $\sharp(T_1 \cap T_2) = d_1 + d_2$. First assume $d_2 \geq 3$. Fix a smooth $T_1 \in |(M_1, 1)|$. Since $d_2 \geq 3$, $(M_2, 1)$ is very ample. Hence Bertini's theorem implies the existence of a smooth $T_2 \in |(M_2, 1)|$ transversal to T_1. Obviously, the same proof works if $d_1 \geq 3$. Now assume $d_1 = d_2 = 2$. There is a finite set $V_i \subset C$ such that $(M_i, 1)$ is very ample outside $U_i := p_1^{-1}(V_i)$, it has no base point and at each point P of U_i the kernel of the differential of the morphism associated to $(M_i, 1)$ is the tangent space of $p_1^{-1}(p_1(P))$ at P.

Indeed, in characteristic $\neq 2$ we have $\sharp(V_i) = 4$, while in characteristic 2 either $\sharp(V_i) = 2$ or $\sharp(V_i) = 1$. Fix M_1 and then take a general M_2. The generality of M_2 implies $V_1 \cap V_2 = \emptyset$. Hence $U_1 \cap U_2 = \emptyset$. Hence a dimensional count gives the existence of smooth T_1, T_2 such that T_2 is transversal to T_1.

Proof of Theorem 1. Use the set-up of Remark 1. Let $u : X = X_1 \cup X_2 \to C$ be a reducible bielliptic curve of type I. Assume the existence of a finite morphism $f : X = X_1 \cup X_2 \to \mathbb{P}^1$ such that the morphism (u, f) is birational onto its image. Set $d_i := \deg(f|X_i)$. Obviously, $d_1 > 0$ and $d_2 > 0$. Since X_1 and X_2 are not rational, $d_1 \geq 2$ and $d_2 \geq 2$. Set $T_i := \mathrm{Im}((u, f)(X_i))$, $i = 1, 2$. T_i is an integral curve contained in W and in a linear system $(M_i, 1)$ for some $M_i \in \mathrm{Pic}^{d_i}(C)$. Since $\sharp(X_1 \cap X_2) = g - 1$, we have $T_1 \cdot T_2 \geq g - 1$, concluding the proof of part (a). Part (b) follows from part (i) of Remark 1. □

Proof of Theorem 2. Take $W' := D \times \mathbb{P}^1$ instead of $W := C \times \mathbb{P}^1$ and copy the proofs of Remark 1 and Theorem 1, except that in part (a) the rationality of X_1 and X_2 only gives the inequality $d_i \geq 1$, $i = 1, 2$. □

Remark 2. Let D' be an integral curve such that $p_a(D') = 1$ and D' has an ordinary cusp. Taking D' instead of either C or D in the proofs of Theorem 1 and 2 we get that the statement of Theorem 2 is true for D'.

Proof of Theorem 3. Set $W := Y \times \mathbb{P}^1$ and adapt Remark 1 and the proof of Theorem 1, because $d_1 + d_2 = (M_1, 1) \cdot (M_2, 1)$. □

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