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NOTE ON TESTING THE EQUALITY OF MEAN COMPONENTS

Yosihito Maruyama

Research Group of Statistical Science
School of Engineering
Osaka Prefecture University
1-1 Gakuen, Naka-ku Sakai-City, Osaka 599-8531, JAPAN
e-mail: maru@ms.osakafu-u.ac.jp

Abstract: The purpose of this note is to express a criterion for testing the equality of the means of several correlated, normally distributed variates in a more directly usable form, and to give a numerical example.

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1. Introduction

A test of the hypothesis that the elements of a multivariate normal mean vector are all equal has been developed in the case of an arbitrary covariance matrix. As a mathematical model for our techniques, we shall assume that the p responses from the j-th sampling unit constitute an observation from a multivariate normal distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)^T$ and some positive definite covariance matrix $\boldsymbol{\Sigma}$ (written $\boldsymbol{\Sigma} > 0$), where $\boldsymbol{\mu}^T$ denotes a transposition of $\boldsymbol{\mu}$, and it is convenient to use the notation $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. A test for the equality of expectations of p corrected normally distributed variates is presented by several ways. Here we shall be mainly concerned with test of the hypothesis $\mathrm{H}_0 : \mu_1 = \cdots = \mu_p$ of equal response means against a general alternative; $\mathrm{H}_1 : \mu_a \neq \mu_b$ for at least one pair (a, b).

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2. Preliminary Results on Matrices

Below we list a number of basic properties in matrix theory, which are used in this note. Most of the omitted proofs are given, for example, in Mardia, Kent and Bibby [2].

Lemma 1. *For* $\boldsymbol{B}\,(p\times n)$ *and* $\boldsymbol{D}\,(n\times p)$*, and non-singular* $\boldsymbol{A}\,(p\times p)$*,*

$$|\boldsymbol{A}+\boldsymbol{BD}| = |\boldsymbol{A}|\cdot\left|\boldsymbol{I}_p+\boldsymbol{A}^{-1}\boldsymbol{BD}\right|.$$

Lemma 2. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be non-singular* $p\times p$ *and* $q\times q$ *matrices, respectively, and let* $\boldsymbol{D}$ *be* $p\times q$ *and* $\boldsymbol{E}$ *be* $q\times p$*. Put* $\boldsymbol{M}:=\boldsymbol{A}+\boldsymbol{DBE}$ *then,*

$$\boldsymbol{M}^{-1}=\boldsymbol{A}^{-1}-\boldsymbol{A}^{-1}\boldsymbol{DB}\left(\boldsymbol{B}+\boldsymbol{BEA}^{-1}\boldsymbol{DB}\right)^{-1}\boldsymbol{BEA}^{-1}.$$

Lemma 3. *Let* $\boldsymbol{C}$ *be a* $(p-1)\times p$ *matrix of rank* $p-1$ *such that* $\boldsymbol{C1}=\boldsymbol{0}$ *where* $\boldsymbol{1}$ *is a p-vector of ones (sometimes* $\boldsymbol{C}$ *is called a contrast matrix). Let* $\boldsymbol{M}$ *be a* $p\times p$ *positive definite matrix. Then,*

$$\boldsymbol{C}^T\left(\boldsymbol{CMC}^T\right)^{-1}\boldsymbol{C}=\boldsymbol{M}^{-1}-\frac{\boldsymbol{M}^{-1}\boldsymbol{11}^T\boldsymbol{M}^{-1}}{\boldsymbol{1}^T\boldsymbol{M}^{-1}\boldsymbol{1}}.$$

3. Testing the Equality of Mean Components

Let $\boldsymbol{x}$ be distributed according to $\mathcal{N}_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$, $\boldsymbol{\Sigma}>0$, and $\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n$ be an independent sample of size n on $\boldsymbol{x}$. The sufficient statistic for $(\boldsymbol{\mu},\boldsymbol{\Sigma})$ is

$$\bar{\boldsymbol{x}}:=\frac{1}{n}\sum_{j=1}^{n}\boldsymbol{x}_j,\qquad \boldsymbol{S}:=\frac{1}{n-1}\sum_{j=1}^{n}\left(\boldsymbol{x}_j-\bar{\boldsymbol{x}}\right)\left(\boldsymbol{x}_j-\bar{\boldsymbol{x}}\right)^T.$$

We are interested in testing the equality of the means μ_i. Thus we wish to test the hypothesis $\mathrm{H}_0:\mu_1=\cdots=\mu_p$, or $\mathrm{H}_0:\boldsymbol{\mu}=k\boldsymbol{1}$, where $\boldsymbol{1}:=(1,\cdots,1)^T$, and k is the unspecified common mean versus $\mathrm{H}_1:\mu_a\neq\mu_b$ for some $a\neq b$, $1\le a<b\le p$. Now let $\boldsymbol{C}$ be a $(p-1)\times p$ matrix of rank $p-1$ such that $\boldsymbol{C1}=\boldsymbol{0}$. Then the hypothesis mentioned above is equivalent to the problem

$$\mathrm{H}_0:\boldsymbol{C\mu}=\boldsymbol{0}\quad\text{versus}\quad\mathrm{H}_1:\boldsymbol{C\mu}\neq\boldsymbol{0}.$$

The likelihood ratio test for this problem is to reject H_0 if

$$\frac{n-p+1}{(n-1)(p-1)}n\left(\boldsymbol{C}\bar{\boldsymbol{x}}\right)^T\left(\boldsymbol{CSC}^T\right)^{-1}\boldsymbol{C}\bar{\boldsymbol{x}}\ge F_{p-1,n-p+1,\alpha},\tag{1}$$

where $F_{p-1,n-p+1,\alpha}$ is the upper $100\alpha\%$ point of the F-distribution with $p-1$ and $n-p+1$ degrees of freedom.

Remark 4. Note that the statistic in (1) does not depend on the choice of $\boldsymbol{C}$ so long as $\boldsymbol{C}\mathbf{1} = \mathbf{0}$. For example, using $\mathrm{H}_0 : \mu_1 - \mu_2 = \cdots = \mu_{p-1} - \mu_p = 0$ leads to

$$\boldsymbol{C}_1\boldsymbol{\mu} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \boldsymbol{\mu} = \mathbf{0},$$

while $\mathrm{H}_0 : \mu_1 - \mu_p = \cdots = \mu_{p-1} - \mu_p = 0$ leads to

$$\boldsymbol{C}_2\boldsymbol{\mu} = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \boldsymbol{\mu} = \mathbf{0}.$$

The $(p-1) \times p$ matrix $\boldsymbol{C}_2$ can be transformed to $\boldsymbol{C}_1$ by subtracting the second row of $\boldsymbol{C}_2$ from the first, the third row from the second, and similarly for the rest, so that $\boldsymbol{C}_1 = \boldsymbol{L}\boldsymbol{C}_2$, where $\boldsymbol{L}$ is a nonsingular matrix of dimension $p-1$; see Rencher [4] for more details.

Moreover, we shall show that construction of the matrix $\boldsymbol{C}$ is unnecessary.

Proposition 5. *The test statistic in (1) is equivalent to*

$$\frac{n-p+1}{p-1}\left(\frac{|\boldsymbol{V}+\boldsymbol{W}|}{|\boldsymbol{V}|}\frac{\mathbf{1}^T(\boldsymbol{V}+\boldsymbol{W})^{-1}\mathbf{1}}{\mathbf{1}^T\boldsymbol{V}^{-1}\mathbf{1}} - 1\right), \tag{2}$$

where $\boldsymbol{V} := (n-1)\boldsymbol{S}$ *and* $\boldsymbol{W} := n\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^T$.

Proof. Put $\boldsymbol{A} = \boldsymbol{V}$, $\boldsymbol{B} = n\bar{\boldsymbol{x}}$ and $\boldsymbol{D} = \bar{\boldsymbol{x}}^T$ in Lemma 1, and further with the aid the identity

$$\left|\boldsymbol{I}_p + \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{D}\right| = \left|\boldsymbol{I}_n + \boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}\right|,$$

then we have

$$\frac{|\boldsymbol{V}+\boldsymbol{W}|}{|\boldsymbol{V}|} = 1 + n\bar{\boldsymbol{x}}^T\boldsymbol{V}^{-1}\bar{\boldsymbol{x}}. \tag{3}$$

Using Lemma 2 with $\boldsymbol{A} = \boldsymbol{V}$, $\boldsymbol{B} = 1$, $\boldsymbol{D} = n\bar{\boldsymbol{x}}$ and $\boldsymbol{E} = \bar{\boldsymbol{x}}^T$, we find that

$$(\boldsymbol{V}+\boldsymbol{W})^{-1} = \boldsymbol{V}^{-1} - \frac{n\boldsymbol{V}^{-1}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^T\boldsymbol{V}^{-1}}{1 + n\bar{\boldsymbol{x}}^T\boldsymbol{V}^{-1}\bar{\boldsymbol{x}}}.$$

Hence,

$$\frac{\mathbf{1}^T(\boldsymbol{V}+\boldsymbol{W})^{-1}\mathbf{1}}{\mathbf{1}^T\boldsymbol{V}^{-1}\mathbf{1}} = 1 - \frac{n\mathbf{1}^T\boldsymbol{V}^{-1}\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}^T\boldsymbol{V}^{-1}\mathbf{1}}{\left(1 + n\bar{\boldsymbol{x}}^T\boldsymbol{V}^{-1}\bar{\boldsymbol{x}}\right)\mathbf{1}^T\boldsymbol{V}^{-1}\mathbf{1}}. \tag{4}$$

From (3) and (4), the second factor in (2) can be written

$$n\bar{\boldsymbol{x}}^T\left(\boldsymbol{V}^{-1}-\frac{\boldsymbol{V}^{-1}\mathbf{1}\mathbf{1}^T\boldsymbol{V}^{-1}}{\mathbf{1}^T\boldsymbol{V}^{-1}\mathbf{1}}\right)\bar{\boldsymbol{x}}, \tag{5}$$

which yields the left-hand side of (1) by Lemma 3. Thus, we complete the proof. □

It should be noted from Proposition 5 that the expression in (2) does not require that a matrix $\boldsymbol{C}$ such that $\boldsymbol{C}\mathbf{1}=\mathbf{0}$ be found.

Remark 6. The test statistic in (1) is also a Hotelling type statistic computed from the transformed observations $\boldsymbol{C}\boldsymbol{x}_1,\cdots,\boldsymbol{C}\boldsymbol{x}_n$. Once again, if $\boldsymbol{C}$ is any $(p-1)\times p$ matrix of rank $p-1$ and is satisfied $\boldsymbol{C}\mathbf{1}=\mathbf{0}$ then

$$n\bar{\boldsymbol{x}}^T\boldsymbol{S}^{-1}\bar{\boldsymbol{x}}-\frac{n\left(\bar{\boldsymbol{x}}^T\boldsymbol{S}^{-1}\mathbf{1}\right)^2}{\mathbf{1}^T\boldsymbol{S}^{-1}\mathbf{1}}=n\left(\boldsymbol{C}\bar{\boldsymbol{x}}\right)^T\left(\boldsymbol{C}\boldsymbol{S}\boldsymbol{C}^T\right)^{-1}\boldsymbol{C}\bar{\boldsymbol{x}}=:T_{\mathrm{H}}, \tag{6}$$

that is essentially equal to (5). The method of transforming the observations and testing the hypothesis that their means are simultaneously zero is due to Hsu [1]. The alternative forms (5) and (6) of the test statistic appear to be useful, particularly if $\boldsymbol{S}^{-1}$ has already been computed for other analysis of the data. For details, see Williams [6] or Srivastava and Khatri [5].

Example 7. We may use the cork data from Table 1 given by Rao [3]. Let x_{N} be the amount of cork in a boring from the north into a cork tree; let x_{E}, x_{S} and x_{W} be defined similarly. The set of amounts in four borings on one tree is considered as an observation from a 4-variate normal distribution. The question is whether the cork trees have the same amount of cork on each side. We make a transformation $y_1:=x_{\mathrm{N}}-x_{\mathrm{E}}+x_{\mathrm{S}}-x_{\mathrm{W}}$, $y_2:=x_{\mathrm{S}}-x_{\mathrm{W}}$ and $y_3:=x_{\mathrm{N}}-x_{\mathrm{S}}$. Further letting $\boldsymbol{x}:=(x_{\mathrm{N}},x_{\mathrm{E}},x_{\mathrm{S}},x_{\mathrm{W}})^T$, $\boldsymbol{y}:=(y_1,y_2,y_3)^T$ and

$$\boldsymbol{C}:=\begin{pmatrix}1&-1&1&-1\\0&0&1&-1\\1&0&-1&0\end{pmatrix},$$

then we have $\boldsymbol{y}=\boldsymbol{C}\boldsymbol{x}$. The data may be summarized as

$$\bar{\boldsymbol{y}}=\begin{pmatrix}8.86\\4.5\\0.86\end{pmatrix},\qquad \boldsymbol{S}_y=\begin{pmatrix}128.72&61.41&-21.02\\61.41&56.93&-28.3\\-21.02&-28.3&63.53\end{pmatrix},$$

which are the sample mean and the sample covariance matrix of the $\boldsymbol{y}$'s, respectively. The value of T_{H} in (6) is 20.736. Hence the test statistic obtained by (1)

$$\frac{28-4+1}{27\times 3}T_{\mathrm{H}}=6.402$$

North (x_{N})	East (x_{E})	South (x_{S})	West (x_{W})
72	66	76	77
91	79	100	75
60	53	66	63
56	68	47	50
56	57	64	58
79	65	70	61
41	29	36	38
81	80	68	58
32	32	35	36
78	55	67	60
30	35	34	26
46	38	37	38
39	39	31	27
39	35	34	37
42	43	31	25
32	30	30	32
37	40	31	25
60	50	67	54
33	29	27	36
35	37	48	39
32	30	34	28
39	36	39	31
63	45	74	63
50	34	37	40
54	46	60	52
43	37	39	50
47	51	52	43
48	54	57	43

Table 1: Weight of cork borings in four directions for 28 trees

is to be compared with the F-significance point with 3 and 25 degrees of freedom. Since $F_{3,25,0.01} = 4.68$, it is significant at the 1% level.

On the other hand, the mean vector for the original observations on 28 trees

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in Table 1 is $\bar{\boldsymbol{x}} = (50.54,\ 46.18,\ 49.68,\ 45.18)^T$, and we have, from Proposition 5

$$\boldsymbol{V} = \begin{pmatrix} 7840.96 & 6041.32 & 7787.82 & 6109.32 \\ 6041.32 & 5938.11 & 6184.61 & 4627.11 \\ 7787.82 & 6184.61 & 9450.11 & 7007.61 \\ 6109.32 & 4627.11 & 7007.61 & 6102.11 \end{pmatrix}.$$

Therefore the statistic in (2) also becomes 6.402, so that the statistic computed from the transformed data is the same as that from the original data. We are able to do test directly and easily without choosing $\boldsymbol{C}$ matrix of contrasts.

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