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A NOTE ON A STATIONARY PROBLEM FOR A BLACK-SCHOLES EQUATION WITH TRANSACTION COSTS

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Abstract: In this paper, we consider the nonlinear Black-Scholes equation arising in certain option pricing models with transaction costs. Following the classical Leland approach and applying Itô's Lemma, the stochastic model yields the nonlinear parabolic partial differential equation for the option price which is denoted by $V(S,t)$,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - a\sigma S^2 \sqrt{\frac{2}{\pi \Delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right| + bS^3 \sigma^2 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 + r\left(S\frac{\partial V}{\partial S} - V \right) = 0.$$

In the spirit of [1], we study the existence of convex solutions of the stationary problem for the above equation with the boundary conditions

$$V(c) = V_c \quad \text{and} \quad V(d) = V_d.$$

We use the method of upper and lower solutions together with Nagumo condition which also yields information on the localization of V.

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1. Introduction

This paper concerns the nonlinear Black-Scholes equation arising in certain option pricing models in presence of transaction costs. The work of Leland [5] is a fundamental reference for this topic, which has attracted the attention of many authors (for example [1, 2, 3, 6]). In their works one can see the study of nonlinear versions of the standard parabolic Black-Scholes PDE, which characterize derivative prices that appear in models with transaction costs. In this paper we follow the framework of Amster and al [1] where the costs are assumed to behave as a nonincreasing linear function of the form $h(\nu) = a - b\nu$, $(a, b > 0)$, depending on the trading stock needed to hedge the replicating portfolio. Following the classical Leland approach and applying Itô's Lemma, the problem yields the nonlinear parabolic partial differential equation for the option price, which is denoted by $V(S,t)$,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - a\sigma S^2 \sqrt{\frac{2}{\pi \Delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right| + bS^3 \sigma^2 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 + r\left(S\frac{\partial V}{\partial S} - V \right) = 0, \quad (1.1)$$

where Δt is the interval between moments of transaction (see [1] for more details).

If $\frac{\partial^2 V}{\partial S^2} > 0$ and a is assumed to be small, the equation (1.1) can be written in the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\widetilde{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + bS^3 \sigma^2 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 + r\left(S\frac{\partial V}{\partial S} - V \right) = 0, \quad (1.2)$$

where $\widetilde{\sigma}^2 = \sigma^2 \left(1 - 2\frac{a}{\sigma}\sqrt{\frac{2}{\pi \Delta t}} \right) > 0$.

We shall study the stationary problem for (1.2)

$$\frac{1}{2}\widetilde{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + bS^3 \sigma^2 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 + r\left(S\frac{\partial V}{\partial S} - V \right) = 0, \quad (1.3)$$

with Dirichlet boundary conditions

$$V(c) = V_c \quad \text{and} \quad V(d) = V_d\,, \quad (1.4)$$

where $0 < c < d$ are fixed real numbers.

In [1] one can find some results concerning problem (1.3)-(1.4). An existence result is proved by studying an associated problem

$$V'' + H\left(S, V, V'\right) = 0, \quad (1.5)$$

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$$V(c) = V_c \quad \text{and} \quad V(d) = V_d, \tag{1.6}$$

and the solution is obtained as the limit of a nonincreasing (respectively nondecreasing) sequence of upper (lower) solutions. It turns out that the function H satisfies Lipschitz conditions with respect to the second variable, and to the third one as well, with Lipschitz constants $K = \frac{2r}{c^2\tilde{\sigma}^2}$ and $K' = \frac{2r}{c\tilde{\sigma}^2}$, respectively.

However, K' is assumed to satisfy

$$K' < \frac{\pi}{d-c},$$

that is,

$$r < \frac{c\tilde{\sigma}^2\pi}{2(d-c)}. \tag{1.7}$$

In this paper we also study the associated problems (1.5)-(1.6) to establish an existence and localization result for problem (1.3)-(1.4) but without imposing condition (1.7). We use the method of upper and lower solutions. In fact, we exhibit a lower solution $\alpha \in C^2$ and an upper solution $\beta \in C^2$ satisfying $\alpha \leq \beta$. Moreover, we show that Nagumo condition is satisfied. Then the result follows by a classical theorem contained in the literature (see [4]).

2. Main Result

We state our main result.

Theorem 1. *Consider the nonlinear Dirichlet boundary value problem:*

$$\begin{cases} b\sigma^2 S^3 (V'')^2 + \frac{1}{2}\tilde{\sigma}^2 S^2 V'' + r(V'S - V) = 0 & \text{in} \quad]c, d[\\ V(c) = V_c \qquad V(d) = V_d. \end{cases} \tag{2.1}$$

The following assertions hold:

1. The function $V(S) = \frac{V_c}{c} S$ is a (linear) solution of the problem (2.1) if and only if $\frac{V_d}{d} = \frac{V_c}{c}$.

2. If $\frac{V_d}{d} < \frac{V_c}{c}$ then the problem (2.1) has a convex solution V such that

$$\frac{V_d}{d} S \leq V(S) \leq \frac{V_d - V_c}{d-c} S + \frac{dV_c - cV_d}{d-c}, \tag{2.2}$$

$$\frac{V_d - V_c}{d-c} < V'(d) \leq \frac{V_d}{d} \text{ and } V'(c) < \frac{V_d - V_c}{d-c}. \tag{2.3}$$

Moreover, in both cases, V is the unique convex solution of (2.1).

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3. Auxiliary Result

Observe that solving algebrically the equation of the problem (2.1) in order of V'', we obtain the equivalent form

$$V'' = \frac{-\tilde{\sigma}^2 \frac{S^2}{2} \pm \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} - 4bS^3\sigma^2 r(V'S - V)}}{2b\sigma^2 S^3}.$$

This fact leads us to consider the auxiliary problem

$$\begin{cases} V'' + g\left(S, V, V'\right) = 0, \\ V\left(c\right) = V_c, \quad V\left(d\right) = V_d, \end{cases} \tag{3.1}$$

where

$$g\left(S, V, V'\right) = \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r\left|V'S - V\right|}}{2b\sigma^2 S^3}.$$

We recall that $\alpha \in C^2$ is a *lower solution* of (3.1) if

$$\begin{cases} \alpha'' + g\left(S, \alpha, \alpha'\right) \geq 0, \\ \alpha\left(c\right) \leq V_c, \quad \alpha\left(d\right) \leq V_d \ . \end{cases} \tag{3.2}$$

Similarly, an *upper solution* $\beta \in C^2$ of (3.1) is defined by reversing the inequalities in (3.2). A *solution* of (3.1) is a function u which is simultaneously a lower and an upper solution. A function f is said to satisfy the Nagumo on some given subset $E \subset I \times \mathbb{R}^2$ if there exists a positive continuous function $\varphi \in C\left(\mathbb{R}_0^+, [\varepsilon, +\infty[\right), \varepsilon > 0$, such that

$$\left|f\left(x, y, z\right)\right| \leq \varphi\left(\left|z\right|\right), \quad \forall\left(x, y, z\right) \in E,$$

and

$$\int_0^{+\infty} \frac{s}{\varphi\left(s\right)} ds = +\infty. \tag{3.3}$$

Now we state an existence and localization result for the problem (3.1).

Theorem 2. *Suppose that*

$$\frac{V_d}{d} \leq \frac{V_c}{c}. \tag{3.4}$$

Then the problem (3.1) *has a solution* V *such that*

$$\frac{V_d}{d} S \leq V(S) \leq \frac{V_d - V_c}{d - c} S + \frac{dV_c - cV_d}{d - c}.$$

Proof. Consider the following functions, defined in $[c, d]$,

$$\alpha\left(S\right) = \frac{V_d}{d} S, \qquad \beta\left(S\right) = \frac{V_d - V_c}{d - c} S + \frac{dV_c - cV_d}{d - c}.$$

Observe that $\alpha\left(\cdot\right)$ and $\beta\left(\cdot\right)$ are lower and upper solutions of (3.1), respectively.

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In fact, we have that

$$\alpha'' + g\left(S,\alpha,\alpha'\right) = 0 + \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r \left|\frac{V_d}{d}\cdot S - \frac{V_d}{d}\cdot S\right|}}{2b\sigma^2 S^3} = \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 0}}{2b\sigma^2 S^3} = 0,$$

and

$$\alpha(c) \le V_c \quad \alpha(d) = V_d.$$

We also have

$$\beta'' + g\left(S,\beta,\beta'\right) = \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r \left|\frac{V_d - V_c}{d-c}S - \frac{V_d - V_c}{d-c}S - \frac{dV_c - cV_d}{d-c}\right|}}{2b\sigma^2 S^3} \le 0$$

and

$$\beta(c) = V_c\,, \quad \beta(d) = V_d.$$

Moreover $\alpha \le \beta$. In fact, algebric computations show easily that

$$\frac{V_d}{d}S \le \frac{V_d - V_c}{d-c}S + \frac{dV_c - cV_d}{d-c}$$

is equivalent to

$$0 \le (d - S)\left(-cV_d + dV_c\right)$$

and then to

$$0 \le \left(-cV_d + dV_c\right)$$

which holds by (3.4).

Now we consider the set:

$$E = \left\{(S,x,y) \in [c,d]\times\mathbb{R}^2 \;\; : \;\; \frac{V_d}{d}S \le x \le \frac{V_d - V_c}{d-c}S + \frac{dV_c - cV_d}{d-c}\right\}.$$

We observe that g satisfies the Nagumo condition in E. In fact,

$$\begin{aligned}
|g(S,x,y)| &= \left|\frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r\,|yS - x|}}{2b\sigma^2 S^3}\right| \\
&\le \frac{\tilde{\sigma}^2 S^2 + \sqrt{4bS^3\sigma^2 r|yS|} + \sqrt{4bS^3\sigma^2 r|x|}}{2b\sigma^2 S^3} \\
&\le \frac{\tilde{\sigma}^2}{2b\sigma^2 c} + \frac{1}{\sigma c\sqrt{c\,b}}\sqrt{r\left(\frac{V_d - V_c}{d-c}d + \frac{dV_c - cV_d}{d-c}\right)} + \frac{\sqrt{r}}{\sigma c\sqrt{b}}\sqrt{|y|},
\end{aligned}$$

that is, for some positive constants k_1, k_2,

$$|g(S,x,y)| \leq k_1 + k_2\sqrt{|y|}.$$

Put $\varphi(y) = k_1 + k_2\sqrt{|y|}$. Then

$$\int_0^{+\infty} \frac{y}{k_1 + k_2\sqrt{|y|}}\,dy = +\infty.$$

Therefore the function g satisfies the Nagumo condition in E. So, by the result contained in [4], we can derive that there exists a solution V_* of (3.1) such that

$$\alpha(S) = \frac{V_d}{d}S \leq V_*(S) \leq \frac{V_d - V_c}{d-c}S + \frac{dV_c - cV_d}{d-c} = \beta(S). \qquad \square$$

4. Proof of the Main Result

Proof of Theorem 1. 1. It is clear that $V(S) = \frac{V_c}{c}S$ satisfies the equation of (2.1) and also the boundary condition $V(c) = V_c$. It is easy to see that V satisfies the boundary condition $V(d) = V_d$ if and only if $\frac{V_d}{d} = \frac{V_c}{c}$, which finishes the proof.

2. Consider a solution of (3.1), whose existence was proved in the previous section, and denote it by V_*. Recall that

$$\alpha(S) = \frac{V_d}{d}S \leq V_*(S) \leq \frac{V_d - V_c}{d-c}S + \frac{dV_c - cV_d}{d-c} = \beta(S).$$

We organize the proof in three steps (that systematize some ideas that can be found in [1]). The first two steps point out two properties of V_* which will be used in Step 3 to conclude that V_* is a solution of (2.1).

Step 1. V_* is convex. Observe that g is a nonpositive function. In fact

$$g\left(S, V_*, V_*'\right) = \frac{\tilde{\sigma}^2\frac{S^2}{2} - \sqrt{\tilde{\sigma}^4\frac{S^4}{4} + 4bS^3\sigma^2 r\left|V_*'S - V_*\right|}}{2b\sigma^2 S^3} \leq \frac{\tilde{\sigma}^2\frac{S^2}{2} - \sqrt{\tilde{\sigma}^4\frac{S^4}{4}}}{2b\sigma^2 S^3} = 0.$$

Therefore, V_* is convex since

$$V_*^{''} = -g\left(S, V_*, V_*'\right) \geq 0.$$

Step 2. The following inequality holds

$$V_*^{'}S - V_* \leq 0.$$

To prove this inequality we will make use of the upper and lower solutions, α and β, of the previous section. Since

$$\alpha(d) = V_*(d) = \beta(d),$$

and for $S \in [c,d]$,

$$\alpha(S) \leq V_*(S) \leq \beta(S),$$

it follows that

$$\beta'(d) \leq V_*'(d) \leq \alpha'(d), \tag{4.1}$$

that is,

$$\frac{V_d - V_c}{d-c} \leq V_*'(d) \leq \frac{V_d}{d}.$$

In particular, we have

$$V_*'(d)\, d \leq V_d = V_*(d). \tag{4.2}$$

As seen in Step 1, V_*'' is nonnegative. Then

$$\left(V_*'(S)\,S - V_*(S)\right)' = V_*''(S)\,S + V_*'(S) - V_*'(S) = V_*''(S)\,S \geq 0$$

and so $V_*'(S)\,S - V_*(S)$ is nondecreasing in S. Therefore, by (4.2),

$$V_*'(S)\,S - V_*(S) \leq V_*'(d)\,d - V_*(d) \leq 0.$$

Step 3. V_* is a solution of the problem (2.1). This statement follows easily from the above comments. In fact, since

$$V_*^{'}S - V_* \leq 0,$$

then

$$\begin{aligned} g\left(S, V_*, V_*'\right) &= \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r \left|V_*'S - V_*\right|}}{2b\sigma^2 S^3} \\ &= \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} - 4bS^3\sigma^2 r \left(V_*'S - V_*\right)}}{2b\sigma^2 S^3}. \end{aligned}$$

So, V_* is a convex solution of

$$V'' = \frac{-\tilde{\sigma}^2 \frac{S^2}{2} + \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} - 4bS^3\sigma^2 r \left(V'S - V\right)}}{2b\sigma^2 S^3},$$

and, therefore, of

$$b\sigma^2 S^3 \left(V''\right)^2 + \frac{1}{2}\tilde{\sigma}^2 S^2 V'' + r\left(V'S - V\right) = 0 \quad \text{in} \quad]c,d[\,,$$

satisfying the boundary conditions

$$V(c) = V_c, \qquad V(d) = V_d$$

and (2.2). As for (2.3), as seen in (4.1)

$$\beta'(d) \leq V_*'(d) \leq \alpha'(d),$$

that is,

$$\frac{V_d - V_c}{d - c} \leq V_*'(d) \leq \frac{V_d}{d}.$$

Since $V_*(S) \leq \beta(S)$ in $[c,d]$ with $V_d = \beta(d)$, the fact that V_* is convex implies that the inequality $\beta'(d) \leq V_*'(d)$ must be strict, that is, $\beta'(d) < V_*'(d)$. Thus

$$\frac{V_d - V_c}{d - c} < V_*'(d) \leq \frac{V_d}{d}.$$

Analogous arguments applied to c show that

$$V_*'(c) < \frac{V_d - V_c}{d - c}.$$

The proof of assertion 2 is finished.

3. The uniqueness result follows immediately from Theorem 2.1 of [1]. $\square$

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