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ON HILBERT TRANSFORM IN CONTEXT OF LOCALLY COMPACT ABELIAN GROUPS

A.R. Mirotin

Department of Mathematics
Skoryna Gomel State University
104, Sovietskaya, Gomel, 246019, BELARUS
e-mail: amirotin@yandex.ru

Abstract: The aim of the paper is to obtain generalizations of M. Riesz's Theorem on the Hilbert transform to compact and locally compact Abelian groups with some fixed subsemigroup of the dual group. Application to generalized Toeplitz operators is given.

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1. Introduction

A celebrated theorem of M. Riesz [11] asserts that the harmonic conjugation map for Fourier series on the circle $\mathbf{T}$ (the Hilbert transform on the circle) is a bounded linear operator on the spaces $L^p(\mathbf{T})$ for $1 < p < \infty$. This result has been extended to compact Abelian groups with ordered duals by S. Bochner [3] and H. Helson [8]. Extensions to locally compact Abelian groups with ordered duals have been obtained by E. Berkson and T.A. Gillespie [2], and E. Hewitt and his collaborators (see Asmar and Hewitt [1] for details).

The aim of the present paper is to obtain generalizations of M. Riesz's Theorem to compact and locally compact Abelian groups with some fixed subsemigroup of the dual group. Application to generalized Toeplitz operators is given.

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In the following, G denotes a locally compact Abelian group with Haar measure m, and X denotes its dual group with Haar measure λ. Measures m and λ are normalized in such a way that the Fourier transform $\mathcal{F}$ which is defined on $L^1(G)$ as

$$\mathcal{F}f(\chi) = \int_G f(x)\overline{\chi(x)}dm(x) \ (\chi \in X),$$

extends from $L^2(G) \cap L^1(G)$ to a unitary operator $\mathcal{F}$ from $L^2(G)$ to $L^2(X)$ (Plancherel's Theorem)).

We assume also that a λ-measurable subsemigroup T of X has been selected in such a way that $T \cap T^{-1} = \{1\}$ if G is compact, and $\lambda(T \cap T^{-1}) = 0$ otherwise.

2. L^2-Hilbert Transform

Consider the following subspace of $L^2(G, \mathbf{C})$ (*the Hardy space, associated with the semigroup* T):

$$H^2_T(G) = \{f \in L^2(G, \mathbf{C}) : \mathcal{F}f = 0 \text{ a.e. on } X \setminus T\}.$$

Lemma 1.

$$\mathrm{Re}H^2_T(G) = \{u \in L^2(G, \mathbf{R}) : \mathcal{F}u = 0 \text{ a.e. on } X \setminus (T \cup T^{-1})\}.$$

Proof. 1) Suppose that $\lambda(T \cap T^{-1}) = 0$. Let $F = \mathcal{F}f$, where $f \in L^2(G, \mathbf{C})$. It is easy to verify that

$$\mathcal{F}(\mathrm{Re}f) = \frac{1}{2}(F + \overline{F}'), \quad \mathcal{F}(\mathrm{Im}f) = -\frac{i}{2}(F - \overline{F}'), \tag{1}$$

where $F'(\chi) := F(\chi^{-1})$. It follows that the left hand side of the equality under consideration is contained in the right hand one. To prove the converse, let u belongs to the right hand side, and $F := 2(\mathcal{F}u)1_T$ (here and below 1_K denotes the characteristic function of the subset $K \subset G$). Then $\overline{F}' = 2(\mathcal{F}u)1_{T^{-1}}$, and in view of the condition 1) we have $\frac{1}{2}(F + \overline{F}') = \mathcal{F}u(1_T + 1_{T^{-1}}) = \mathcal{F}u$. Now if we put $f := \mathcal{F}^{-1}F$, then $f \in H^2_T(G)$, and hence by (1) we have $\mathcal{F}(\mathrm{Re}f) = \frac{1}{2}(F + \overline{F}') = \mathcal{F}u$, i.e. $u = \mathrm{Re}f$.

2) Now suppose that G is compact and $T \cap T^{-1} = \{1\}$. The fact that the left hand side of the equality being proved is contained in the right hand one can be seen as above. Now let u belongs to the right hand side of the equality. Then the function $F := (\mathcal{F}u)1_{T\setminus\{1\}}$ belongs to $L^2(X)$, and therefore the function $f := \mathcal{F}u(1)1 + 2\mathcal{F}^{-1}F$ belongs to $L^2(G)$. Moreover, $\mathcal{F}f(\chi) = \mathcal{F}u(1)1_{\{1\}}(\chi) + 2\mathcal{F}u(\chi)1_{T\setminus\{1\}}(\chi) = 2\mathcal{F}u(\chi)$ for $\chi \in T \setminus \{1\}$, $= \mathcal{F}u(1)$ for $\chi = 1$,

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and $= 0$ for $\chi \notin T$. Consequently, $f \in H^2_T(G)$. Taking into account that u is real-valued we deduce that

$$\mathcal{F}(\mathrm{Re} f) = \frac{1}{2}(\mathcal{F}f + \overline{(\mathcal{F}f)'}) = \mathcal{F}u,$$

and so $u = \mathrm{Re} f$. □

Lemma 2. *Suppose that G is noncompact (and $\lambda(T \cap T^{-1}) = 0$). Then for every $u \in \mathrm{Re}H^2_T(G)$ there exists a unique function $v \in \mathrm{Re}H^2_T(G)$ such that $u + iv \in H^2_T(G)$.*

Proof. Let the function $f \in H^2_T(G)$ has the form $f = u + iv$. Then $v = \mathrm{Re}(-if) \in \mathrm{Re}H^2_T(G)$. Formula (1) yields $\mathcal{F}u = \frac{1}{2}(F + \overline{F}')$, where the function $F = \mathcal{F}f$ equals to 0 a.e. on $X \setminus T$. Since $\lambda(T \cap T^{-1}) = 0$, it follows that $F = 2(\mathcal{F}u)1_T$. The last equality implies that the function u uniquely determines F, f, and v. □

Lemma 3. *Suppose that G is compact (and $T \cap T^{-1} = \{1\}$). Then for every $u \in \mathrm{Re}H^2_T(G)$ there exists a unique function $v \in \mathrm{Re}H^2_T(G)$ such that $\mathcal{F}v(1) = 0$ and $u + iv \in H^2_T(G)$.*

Proof. Let f, v, and F be as in the proof of the previous Lemma. The first formula in (1) implies $F(\chi) = 2(\mathcal{F}u)(\chi)1_T(\chi)$ for $\chi \in T \setminus \{1\}$. Therefore u uniquely determines F on the set $X \setminus \{1\}$. Now the second formula in (1) shows, that u uniquely determines $\mathcal{F}v$ on the set $X \setminus \{1\}$. □

Definition 4. In both cases considered in Lemmas 2 or 3 the function v is called *harmonic conjugate* of u.

Let

$$L^2_T(G) := \{u \in L^2(G, \mathbf{C}) : \mathcal{F}u = 0 \text{ a.e. on } X \setminus (T \cup T^{-1})\}$$

(in other words $L^2_T(G)$ is a complexification of $\mathrm{Re}H^2_T(G)$). The linear map $\mathcal{H}$, which is a linear continuation of $u \mapsto v$ to $L^2_T(G)$ is called *L^2-Hilbert transform*.

Consider the following function on X

$$\mathrm{sgn}_T := 1_T - 1_{T^{-1}}.$$

The next formula will be very useful.

Lemma 5. *Let $u \in L^2_T(G)$ and $v = \mathcal{H}u$. In both cases considered in Lemmas 2 or 3 the following equality holds:*

$$\mathcal{F}v = -i\mathrm{sgn}_T\mathcal{F}u. \tag{2}$$

Proof. It is sufficient to consider the case $u \in \mathrm{Re}H^2_T(G)$. Choose $f \in H^2_T(G)$ such that $f = u + iv$, and let $F = \mathcal{F}f$. It is straightforward to check by means of (1) that (2) holds for $\chi \in T \setminus T^{-1}$ and for $\chi \in T^{-1} \setminus T$. It follows that the condition $\lambda(T \cap T^{-1}) = 0$ implies (2) for a.e. $\chi \in X$.

Now let X be discrete and $T \cap T^{-1} = \{1\}$. Then (2) holds for all $\chi \in X$ in virtue of $\mathcal{F}v(1) = 0$. □

The following theorem contains important properties of L^2-Hilbert transform in the case considered in Lemma 2.

Theorem 6. *Suppose that G is non compact. The operator $\mathcal{H}$ in $L^2_T(G)$ possesses the following properties (I is the identity operator in $L^2_T(G)$):*

(i) $\mathcal{H}^* = -\mathcal{H}$ (antisymmetry);

(ii) $\mathcal{H}^2 = -I$, i.e. $\mathcal{H}^{-1} = -\mathcal{H}$ (inverse formula);

(iii) property of being unitary.

Proof. (i) Using the Plancherel's Theorem and Lemma 5, we obtain for every $u_1, u_2 \in L^2_T(G)$ (brackets below denote inner products in $L^2(G)$ and $L^2(X)$)

$$\langle u_1, \mathcal{H}u_2 \rangle = \langle \mathcal{F}u_1, \mathcal{F}\mathcal{H}u_2 \rangle = \int\limits_X \mathcal{F}u_1 \overline{(-i)\mathrm{sgn}_T \mathcal{F}u_2} d\chi$$
$$= -\langle \mathcal{F}\mathcal{H}u_1, \mathcal{F}u_2 \rangle = -\langle \mathcal{H}u_1, u_2 \rangle.$$

(ii) Lemma 2 states that for every $u \in \mathrm{Re}H^2_T(G)$ there exists unique $v \in \mathrm{Re}H^2_T(G)$, such that $f := u + iv \in H^2_T(G)$. Since by definition $\mathcal{H}u = v$, and $-if = v - iu$, then (again by definition) $\mathcal{H}v = -u$. It follows that the first equality in (ii) holds for the restriction $\mathcal{H}|\mathrm{Re}H^2_T(G)$. Now by linearity it is true for the whole space $L^2_T(G)$. In turn, this implies the bijectivity of $\mathcal{H}$, and the inverse formula follows.

(iii) It is immediate from (i) and (ii). □

Examples 7. 1) (The Hilbert transform on $\mathbf{R}$) Let $G = \mathbf{R}$ ($X = \mathbf{R}$), and $T = \mathbf{R}_+$. Since $T \cup (-T) = \mathbf{R}$, we have $L^2_T(\mathbf{R}) = L^2(\mathbf{R})$. Let $H^2(E_2^+)$ denotes the Hardy space in the upper half-plane E_2^+. It is well known (see, e. g., Stein and Weiss [13], Chapter III) that $H^2_{\mathbf{R}_+}(\mathbf{R})$ is the space of boundary values for functions in $H^2(E_2^+)$ and is canonically isomorphic to $H^2(E_2^+)$. Here the fact that $\mathcal{H} : u \mapsto v$ where $u \in L^2(\mathbf{R})$ means that $u + iv$ is a boundary value of a function from $H^2(E_2^+)$. One can show (see Asmar and Hewitt [1]) that

$$\mathcal{H}f(t) = \frac{1}{\pi} \int\limits_{\mathbf{R}} \frac{f(x)}{t - x} dx$$

(the Cauchy principal value integral).

2) (The Hilbert transform on $\mathbf{R}^n$ associated with a cone) Let $G = \mathbf{R}^n$ ($X = \mathbf{R}^n$), and K be an open acute cone in G with the conjugate one K^*. Let T_K be the tube domain in $\mathbf{C}^n$ associated with K. The Hardy space $H^2(T_K)$ consists of such analytic functions F in the interior of T_K that

$$\sup_{y\in K}\int\limits_{\mathbf{R}^n}|F(x+iy|^2dx<\infty.$$

It is well known (see, e.g., Stein and Weiss [13], Chapter III), that every function $F \in H^2(T_K)$ has the boundary value function $F \in H^2_{K^*}(\mathbf{R}^n)$, such that $F(x+iy) \to F(x)$ in L^2-norm ($y \in K, y \to 0$). Moreover, the map $F(z) \mapsto F(x)$ is an isomorphism of corresponding spaces. In the case when $G = \mathbf{R}^n$, and $T = K^*$, functions u and $\mathcal{H}u$ are real and imaginary parts respectively of the boundary value of some function in $H^2(T_K)$.

Consider the special case where $n = 2$, and $K = (0,\infty)^2$ (then $K^* = \mathbf{R}^2_+$). In this case we have for $f \in L^2_{\mathbf{R}^2_+}(\mathbf{R}^2)$

$$\mathcal{H}f(t_1,t_2)=\frac{1}{2\pi}\left(\int\limits_{\mathbf{R}}\frac{f(t_1,x_2)}{t_2-x_2}dx_2+\int\limits_{\mathbf{R}}\frac{f(x_1,t_2)}{t_1-x_1}dx_1\right). \tag{3}$$

In other words,

$$\mathcal{H}f(t_1,t_2)=\frac{1}{2}(\mathcal{H}_{x_2\to t_2}f(t_1,x_2)+\mathcal{H}_{x_1\to t_1}f(x_1,t_2)).$$

To prove (3), we shall verify the property of Hilbert transform, given by Lemma 5. According to the above mentioned lemma we have (below $\mathcal{F}$ denotes the two-dimensional Fourier transform, and $\mathcal{F}$ endowed with indices — the one-dimensional ones)

$$\begin{aligned}\mathcal{F}(\mathcal{H}_{x_1\to t_1}f(x_1,t_2))(\lambda_1,\lambda_2) &= \mathcal{F}_{t_2\to\lambda_2}(\mathcal{F}_{t_1\to\lambda_1}\mathcal{H}_{x_1\to t_1}f(x_1,t_2))\\ &= \mathcal{F}_{t_2\to\lambda_2}(-i\mathrm{sgn}\lambda_1\mathcal{F}_{t_1\to\lambda_1}f(t_1,t_2)) = -i\mathrm{sgn}\lambda_1\mathcal{F}f(\lambda_1,\lambda_2),\end{aligned}$$

and similarly,

$$\mathcal{F}(\mathcal{H}_{x_2\to t_2}f(t_1,x_2))(\lambda_1,\lambda_2)=-i\mathrm{sgn}\lambda_2\mathcal{F}f(\lambda_1,\lambda_2).$$

It remains to note that $\mathrm{sgn}\lambda_1+\mathrm{sgn}\lambda_2 = 2\mathrm{sgn}_{\mathbf{R}^2_+}(\lambda_1,\lambda_2)$.

The inverse formula shows that for every $g \in L^2_{\mathbf{R}^2_+}(\mathbf{R}^2)$ the equation

$$\frac{1}{2\pi}\left(\int\limits_{\mathbf{R}}\frac{f(t_1,x_2)}{t_2-x_2}dx_2+\int\limits_{\mathbf{R}}\frac{f(x_1,t_2)}{t_1-x_1}dx_1\right)=g(t_1,t_2)$$

has the unique solution $f \in L^2_{\mathbf{R}^2_+}(\mathbf{R}^2)$, given by the equality

$$f(x_1, x_2) = \frac{1}{2\pi}\left(\int_{\mathbf{R}} \frac{g(x_1, t_2)}{t_2 - x_2} dt_2 + \int_{\mathbf{R}} \frac{g(t_1, x_2)}{t_1 - x_1} dt_1\right).$$

Consider the properties of L^2-Hilbert transform in the compact case (see Lemma 3).

Theorem 8. *Suppose that G is compact. The operator $\mathcal{H}$ in $L^2_T(G)$ possesses the following properties:*

(i) $\mathcal{H}^* = -\mathcal{H}$ (antisymmetry);

(ii) $\mathcal{H}^2 f = -f + \mathcal{F}f(1)$, i.e. the following inverse formula holds: if $g = \mathcal{H}f$, then $f = -\mathcal{H}g + \mathcal{F}f(1)$ $(f, g \in L^2_T(G))$;

(iii) $\|\mathcal{H}f\|^2 = \|f\|^2 - |\mathcal{F}f(1)|^2$.

Proof. (i) One can to prove (i) in the same manner as in Theorem 6.

(ii) It is sufficient to consider the case $u \in \mathrm{Re}H^2_T(G)$. First note that $1 \in \mathrm{Re}H^2_T(G)$, and, by definition, $\mathcal{H}1 = 0$. So $\mathcal{H}c = 0$ for every complex number c. Further, let $u_1 = u - \mathcal{F}u(1)$ for some $u \in \mathrm{Re}H^2_T(G)$. Then $\mathcal{F}u_1(1) = 0$. If $f \in H^2_T(G)$ and $u = \mathrm{Re}f$, then $u_1 = \mathrm{Re}(f - \mathcal{F}u(1)) \in \mathrm{Re}H^2_T(G)$, because $\mathcal{F}u(1) \in \mathbf{R}$. If $v = \mathcal{H}u_1$, then $f_1 = u_1 + iv \in H^2_T(G)$. But $-if_1 = v - iu_1$, and so, by definition, $u_1 = \mathcal{H}v$. The last two equalities for $\mathcal{H}$ imply (ii) with u instead of f (to prove the inverse formula, one should to apply the Hilbert transform to the equality $g = \mathcal{H}f$).

(iii) The Plancherel Theorem and Lemma 5 entail that $(f \in L^2_T(G))$

$$\|\mathcal{H}f\|^2 = \|\mathcal{F}\mathcal{H}f\|^2 = \|-i\mathrm{sgn}_T\mathcal{F}f\|^2 = \sum_{\chi\in X} \mathrm{sgn}^2_T(\chi)|\mathcal{F}f(\chi)|^2$$

$$= \sum_{\chi\in X} |\mathcal{F}f(\chi)|^2 - |\mathcal{F}f(1)|^2 = \|\mathcal{F}f\|^2 - |\mathcal{F}f(1)|^2 = \|f\|^2 - |\mathcal{F}f(1)|^2. \quad \square$$

Corollary 9. *Let G be compact and*

$$L^2_{T0}(G) = \{f \in L^2(G) : \mathcal{F}f = 0 \text{ on } X \setminus (T \cup T^{-1}), \text{ and } \mathcal{F}f(1) = 0\}.$$

Then the restriction $\mathcal{H}_0 := \mathcal{H}|L^2_{T0}(G)$ possesses the following properties:

(i) $\mathcal{H}^*_0 = -\mathcal{H}_0$ (antisymmetry);

(ii) $\mathcal{H}^{-1}_0 = -\mathcal{H}_0$ (inverse formula);

(iii) it is unitary.

Examples 10. 1) (The Hilbert transform on $\mathbf{T}$) Let $G = \mathbf{T}$ ($X = \mathbf{Z}$), and $T = \mathbf{Z}_+$. Since $T \cup (-T) = \mathbf{Z}$, $L^2_{\mathbf{Z}_+}(\mathbf{T}) = L^2(\mathbf{T})$. It is well known (see, e.g., Garnett [7], Chapter II), that every function F in the Hardy class $H^2(\mathbf{D})$ has the boundary value function $F \in H^2_{\mathbf{Z}_+}(\mathbf{T})$, such that $F(re^{it}) \to F(e^{it})$ in L^2-norm ($r \to 1$). Moreover, the map $F(z) \mapsto F(e^{it})$ is an isomorphism of corresponding spaces. Here the equality $v = \mathcal{H}u$ ($u \in L^2(\mathbf{T})$) means that $u + iv$ is the boundary value function of some function in $H^2(\mathbf{D})$, and $\mathcal{F}v(0) = 0$. As in the example 7, 1), one can prove the following integral representation:

$$\mathcal{H}f(t) = \frac{1}{2\pi} \int\limits_{[0,2\pi]} f(\theta) \cot \frac{t-\theta}{2} d\theta,$$

(the Cauchy principal value integral).

2) (The Hilbert transform on $\mathbf{T}^n$) Let $G = \mathbf{T}^n$ ($X = \mathbf{Z}^n$), and $T = \mathbf{Z}^n_+$. For each $F(z)$ from $H^2(\mathbf{D}^n)$ the L^2-limit $F(t) := \lim_{r \to 1-0} F(rt)$ exists, and the mapping $F(z) \mapsto F(t)$ is an isomorphism of Hilbert spaces $H^2(\mathbf{D}^n)$ and $H^2_{\mathbf{Z}^n_+}(\mathbf{T}^n)$ (Rudin [12], 3.4). Here the equality $v = \mathcal{H}u$ where $u \in L^2_{\mathbf{Z}^n_+}(\mathbf{T}^n)$, means that $u + iv$ is the boundary value of some function from $H^2(\mathbf{D}^n)$, and $\mathcal{F}v(0) = 0$.

Now let $n = 2$. As in Example 7, 2) one can get the following integral representation ($f \in L^2_{\mathbf{Z}^2_+}(\mathbf{T}^2)$)

$$\mathcal{H}f(t_1, t_2) = \frac{1}{4\pi} \Big(\int\limits_{[0,2\pi]} f(t_1, \theta_2) \cot \frac{t_2 - \theta_2}{2} d\theta_2 + \int\limits_{[0,2\pi]} f(\theta_1, t_2) \cot \frac{t_1 - \theta_1}{2} d\theta_1 \Big).$$

3. L^p-Hilbert Transform

In this section, the case of L^p-spaces for $p \neq 2$ is examined.

Following Bourbaki [5], we denote by $A(G)$ the vector subspace in $L^1(G)$, generated by the set of functions of the form $f * g$ where $f, g \in L^1(G) \cap L^2(G)$.

Lemma 11. *(see Bourbaki [5], Lemma 2.1.1) (i) The space $A(G)$ is an ideal of the algebra $L^1(G)$, and $A(G) \subset L^1(G) \cap L^2(G)$.*

(ii) Let $p \in (1, +\infty)$, $f \in L^p(G)$, $\epsilon > 0$. For every neighborhood V of identity in G there is such function $\phi \in A(G)$ with compact support in V, that $\|\phi * f - f\|_p < \epsilon$.

Let $B(G)$ be the vector space of functions in $L^1(G)$, whose Fourier transforms belong to $L^1(X)$. It is clear that $A(G) \subset B(G)$.

Lemma 12. *(i) The Fourier transform is a bijection of $B(G)$ onto $B(X)$.*
(ii) $B(G)$ is dense in $L^p(G)$ for every $p \in (1, +\infty)$.

Proof. For (i) see Bourbaki [5], Theorem 2.1.3.

Since the space $C_{00}(G)$ of continuous functions with compact support is dense in $L^p(G)$, (ii) follows from preceding Lemma. □

Definition 13. Let $B_T(G)$ denotes the vector space of functions in $B(G)$ whose Fourier transforms vanish λ-a.e. on $X \setminus (T \cup T^{-1})$. By $L^p_T(G)$ we shall denote the (closed) vector subspace of $L^p(G)$, generated by $B_T(G)$.

Theorem 14. *Suppose that G is noncompact. For every $p \in \{n, n/n-1 : n = 2, 3, \ldots\}$ the Hilbert transform extends from $B_T(G)$ to linear bounded operator $\mathcal{H}$ from $L^p_T(G)$ to $L^p(G)$. If, in addition, $\mathrm{int}T \neq \emptyset$ and $\lambda(\partial T) = 0$, then $\mathcal{H}$ acts in $L^p_T(G)$.*

Proof. First consider the case $p \in \{2, 3, \ldots\}$. Let $b \in B_T(G)$ be real-valued, and $f := b + i\mathcal{H}b$, $(f \in H^2_T(G))$, $F := \mathcal{F}f$. Then $F = (1 + \mathrm{sgn}_T)\mathcal{F}b = 2\mathcal{F}b$ λ-a.e. on T and $= 0$ λ-a.e. on $X \setminus T$. Assume that $F(\chi) = 0$ for all $\chi \in X \setminus T$. Since $F \in L^1(G) \cap L^2(G)$, F^{*p} belongs to $A(G)$. So $\mathcal{F}f^p = F^{*p} \in B(X)$, and $f^p \in B(G)$ $(p \geq 2)$. Note that $F^{*p}(\chi) = 0$ for all $\chi \in X \setminus T$. Indeed, for such χ we have

$$F^{*2}(\chi) = \int_T F(\chi\xi^{-1})F(\xi)d\xi = 0,$$

since $\chi\xi^{-1} \notin T$ for $\chi \notin T$, $\xi \in T$, and the claim follows by induction. The function $F^{*p} \in \mathcal{F}L^1(G)$ is continuous, and so $F^{*p} = 0$ on the closure of $X \setminus T$. But $\lambda(T \cap T^{-1}) = 0$ implies $1 \in X \setminus \mathrm{int}T = \overline{X \setminus T}$. Thus

$$\mathcal{F}f^p(1) = \int_G (b + i\mathcal{H}b)^p dm = 0. \tag{4}$$

Note that, by definition, $\mathcal{H}b$ is real-valued. Putting in (4) $p = 2l + 1$ $(l \in \mathbf{N})$ and taking the imaginary part, one can get using M. Riesz's arguments (see, e.g., Dunford and Schwartz [6], XI.7.8) that the Hilbert transform extends from $B_T(G)$ to linear bounded operator $\mathcal{H}$ from $L^p_T(G)$ to $L^p(G)$. The same is true for even p, so $\|\mathcal{H}f\|_p \leq K_p\|f\|_p$ for $f \in L^p_T(G)$ for some constant K_p.

Now let $q \in \{2, 3, \ldots\}$, $p^{-1} + q^{-1} = 1$, and $f \in B_T(G)$. By the Plancherel Theorem

$$\|\mathcal{H}f\|_p = \sup\left\{\left|\int_G \mathcal{H}f\overline{g}dm\right| : g \in B(G), \|g\|_q \leq 1\right\}$$

$$= \sup\left\{\left|\int_X \mathcal{F}(\mathcal{H}f)\overline{\mathcal{F}g}d\lambda\right| : g \in B(G), \|g\|_q \leq 1\right\}$$
$$\leq \sup\left\{\left|\int_X \mathcal{F}(\mathcal{H}f)1_{T\cup T^{-1}}\overline{\mathcal{F}g}d\lambda\right| : g \in L^2(G), \|g\|_q \leq 1\right\}.$$

Since (by the same theorem)

$$\{\mathcal{F}g1_{T\cup T^{-1}} : g \in L^2(G)\} = \{\mathcal{F}g1_{T\cup T^{-1}} : g \in L^2_T(G)\},$$

we have in virtue of $\mathcal{H}^*g = -\mathcal{H}g$ and the Hölder inequality that

$$\|\mathcal{H}f\|_p \leq \sup\left\{\left|\int_X \mathcal{F}(\mathcal{H}f)\overline{\mathcal{F}g}d\lambda\right| : g \in L^2_T(G), \|g\|_q = 1\right\}$$
$$\sup\left\{\left|\int_G \mathcal{H}f\overline{g}dm\right| : g \in L^2_T(G), \|g\|_q \leq 1\right\}$$
$$= \sup\{|\langle f, \mathcal{H}g\rangle| : g \in L^2_T(G), \|g\|_q \leq 1\}$$
$$\leq \sup\{\|f\|_p\|\mathcal{H}g\|_q : g \in L^2_T(G), \|g\|_q \leq 1\} \leq K_q\|f\|_p.$$

Now let, in addition, $\mathrm{int}T \neq \emptyset$ and $\lambda(\partial T) = 0$. We shall show that $\mathcal{H}b \in L^p_T(G)$ for $b \in B_T(G)$. The function

$$\psi := \mathcal{F}\mathcal{H}b = -i\mathrm{sgn}_T\mathcal{F}b$$

belongs to $L^q(X)$ and vanishes on $X \setminus (T \cup T^{-1})$. Since $\lambda(\partial(T \cup T^{-1})) = 0$, for every $\epsilon > 0$ there is such $\omega \in C_{00}(X)$ supported on $\mathrm{int}(T \cup T^{-1})$, that $\|\psi - \omega\|_q < \epsilon/2$. Choose a neighborhood of identity $V \subset X$ such that the closure of $V\mathrm{supp}(\omega)$ contains in $\mathrm{int}(T \cup T^{-1})$, and a function $\phi \in A(X)$ with the properties $\mathrm{supp}(\phi) \subset V$, and $\|\phi*\omega-\omega\|_q < \epsilon/2$. Then the function $\alpha := \phi*\omega$ belongs to $A(X)$, and its support $\mathrm{supp}(\alpha)$ contains in the closure of $V\mathrm{supp}(\omega)$. Therefore $\alpha = \mathcal{F}a$ for some $a \in B_T(G)$, and the Hausdorff-Young inequality implies that

$$\|\mathcal{H}b - a\|_p \leq \|\psi - \alpha\|_q < \epsilon. \qquad \square$$

Consider conditions for validity of the conclusion of the previous theorem for all real $p > 1$.

Definition 15. Let $p > 1$. We shall call the pair (G, T) *p-interpolational*, if the space $L^p_T(G)$ is linear interpolational between $L^{p_0}_T(G)$ and $L^{p_1}_T(G)$ for some p_0, p_1 from $\{n, n/n-1 : n = 2, 3, \ldots\}$, such that $p_0 \leq p \leq p_1$.

In the case $T \cup T^{-1} = G$ the pair (G, T) is p-interpolational for every $p > 1$ by the Riesz-Thorin Theorem. The pair $(\mathbf{T}^2, \mathbf{Z}_+^2)$ is p-interpolational for every $p > 1$, too (Kislyakov and Kuan-hua-Shu [9]). In general, the problem of description of p-interpolational pairs seems to be open.

The following theorem is now obvious.

Theorem 16. *Suppose that G is noncompact, and the pair (G, T) is p-interpolational for some $p > 1$. Then the Hilbert transform extends from $B_T(G)$ to linear bounded operator $\mathcal{H}$ from $L^p_T(G)$ to $L^p(G)$. If, in addition,* $\mathrm{int}T \neq \emptyset$ *and $\lambda(\partial T) = 0$, then $\mathcal{H}$ acts in $L^p_T(G)$.*

Corollary 17. *Suppose that G is noncompact, and $T \cup T^{-1} = X$. Then for every $p \in (1, \infty)$ the Hilbert transform extends from $B(G)$ to linear bounded operator $\mathcal{H}$ in $L^p(G)$.*

We now turn to the compact case.

Theorem 18. *Suppose that G is compact. For every $p \in \{n, n/n-1 : n = 2, 3, \dots\}$ the Hilbert transform extends from $B_T(G)$ to linear bounded operator $\mathcal{H}$ from $L^p_T(G)$ to $L^p(G)$.*

Proof. Let b, f, F be the same as in the proof of the theorem 14, and $p \in \{2, 3, \dots\}$. The function $F = (1 + \mathrm{sgn}_T)\mathcal{F}b$ vanishes on $X \setminus T$ and T^{-1}. Using induction and the formula

$$F^{*(p+1)}(\chi) = \sum_{\xi \in X} F^{*p}(\chi\xi^{-1})F(\xi)$$

one can easily prove that F^{*p} vanishes on $X \setminus T$ and T^{-1}, too. In particular, $F^{*p}(1) = 0$. The rest of the proof is the same as ones of Theorem 14. □

The following analogs of previous results obtained for noncompact groups also hold.

Theorem 19. *Suppose that G is compact, and the pair (G, T) is p-interpolational for some $p > 1$. Then the Hilbert transform extends from $B_T(G)$ to linear bounded operator $\mathcal{H}$ from $L^p_T(G)$ to $L^p(G)$.*

Corollary 20. *Suppose that G is compact, and $T \cup T^{-1} = X$. Then for every $p \in (1, \infty)$ the Hilbert transform extends from $B(G)$ to linear bounded operator $\mathcal{H}$ in $L^p(G)$.*

4. Application: Generalized Toeplitz Operators

For classical theory of Toeplitz operators see, e.g., Böttcher and Silberman [4]. Here we consider the generalization of this notion to compact Abelian groups (see Murphy [10] for the case of $H^2_T(G)$).

Definition 21. Define the *Riesz projection* P_+ by

$$P_+f = \mathcal{F}^{-1}(1_T\mathcal{F}f), \quad f \in L^2(G).$$

It is easy to check that P_+ is an orthogonal projection $L^2(G) \to H^2_T(G)$.

Lemma 22. *(i) For every $f \in L^2_T(G)$*

$$P_+f = \begin{cases} \frac{1}{2}(f + i\mathcal{H}f), & G \text{ is noncompact}, \\ \frac{1}{2}(f + 2\mathcal{F}f(1)1 + i\mathcal{H}f), & G \text{ is compact}. \end{cases}$$

(ii) If the Hilbert transform $\mathcal{H}$ extends from $B_T(G)$ to linear bounded operator in $L^p_T(G)$ $(p > 1)$, the same is true for P_+.

Proof. (i) Consider the operator $J : L^2(G) \to L^2_T(G)$,

$$Jf = \mathcal{F}^{-1}((1_T + 1_{T^{-1}})\mathcal{F}f).$$

Since $1_T + 1_{T^{-1}} = \text{sgn}^2_T + 21_{\{1\}}$ for compact G, and $= \text{sgn}^2_T$ for noncompact G, we have in view of Lemma 5 and Theorem 6 ($f \in L^2_T(G)$)

$$Jf = \mathcal{F}^{-1}(\text{sgn}_T(\text{sgn}_T\mathcal{F}f) + 21_{\{1\}}\mathcal{F}f) = -\mathcal{H}^2f + 2\mathcal{F}f(1)1 = f + 2\mathcal{F}f(1)1$$

for compact G, and $Jf = f$ for noncompact G.

On the other hand,

$$Jf + (-\frac{1}{i})\mathcal{H} = 2\mathcal{F}^{-1}(1_T\mathcal{F}f) = 2P_+f.$$

(ii) It follows from (i) (and Hölder inequality in the compact case). □

Definition 23. Suppose that G is compact. Let $\varphi \in L^\infty(G)$. An operator T_φ on $B_T(G)$ defined by

$$T_\varphi f = P_+(\varphi f), \quad f \in B_T(G)$$

is called *Toeplitz operator* (with symbol φ).

Let $p > 1$ and suppose that G is compact. It follows from Lemma 22 that if the Hilbert transform $\mathcal{H}$ extends from $B_T(G)$ to linear bounded operator in $L^p_T(G)$, then T_φ is an L^p-bounded operator on $B_T(G)$ and hence it can be extended to bounded linear operator on $H^p_T(G)$.

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