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TSALLIS RELATIVE ENTROPY FOR CONVEX FUNCTIONALS

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Abstract: The fundamental goal of this article is to introduce an extension of the Tsallis relative entropy,

$$T_p(A/B) = \frac{A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^p A^{1/2} - A}{p},$$

from positive operators to convex functionals. Several results together with their proofs, already stated for relative operator entropy, are here obtained immediately in a fast way.

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1. Introduction

In previous work [7], we have introduced an extension of the relative operator entropy,

$$S(A/B) := A^{1/2} Log\left(A^{-1/2}BA^{-1/2}\right) A^{1/2},$$

when the positive operator variables A and B are convex functionals. This functional approach, stated under a convex character, has allowed us to simplify various proofs differently discussed in several papers by many authors. In the

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present work, we shall go in the same sense and we extend another notion, namely the Tsallis relative operator entropy $T_p(A/B)$, from positive operators to convex functionals which we call parameterized functional entropy. The central definition of this extension, always having a convex character, permits us to easily study the corresponding properties and to prove the associate results. When the considering convex functionals are quadratic associated respectively to positive operators A and B, we obtain in a fast way that of relative operator entropy $T_p(A/B)$, already introduced in the literature (see [4] for instance). Afterwards, we establish that the relative functional entropy,

$$\mathcal{S}(f/g) = \int_0^1 \frac{((1-t).f^* + t.g^*)^* - f}{t} dt,$$

introduced by the author [7], is the limit of our parameterized functional entropy, when the parameter p goes to 0. In the quadratical case, we obtain immediately the next known relation

$$\lim_{p \downarrow 0} T_p(A/B) = S(A/B),$$

differently proved in the literature.

2. Basic Notions

In this section, we recall some basic notions about convex analysis needed throughout the paper. For further details, we refer the reader to [1], [2], [8] for example. Let E be a real locally convex space, E^* its topological dual, and $< .,. >$ the duality bracket between E and E^*. If we denote by $\overline{I\!R}^E$ the space of all functions defined from E into $\overline{I\!R} := I\!R \cup \{-\infty, +\infty\}$, we can extend the structure of $I\!R$ on $\overline{I\!R}$ by setting

$$\forall \ x \in \overline{I\!R}, \quad -\infty \leq x \leq +\infty, \quad (+\infty) + x = +\infty, \quad 0.(+\infty) = +\infty.$$

The space $\overline{I\!R}^E$ is equipped with the point-wise partial ordering defined by

$$\forall \ f, \ g \in \overline{I\!R}^E, \quad f \leq g \iff \forall u \in E \quad f(u) \leq g(u).$$

Given a functional $f : E \to \widetilde{I\!R} := I\!R \cup \{+\infty\}$, the Legendre-Fenchel conjugate of f is the functional $f^* : E^* \longrightarrow \overline{I\!R}$ such that

$$\forall u^* \in E^* \quad f^*(u^*) = \sup_{u \in E} \{< u, u^* > - f(u)\}.$$

It is obvious that, if $f \leq g$ then $g^* \leq f^*$.

We notice that, if E is a complex space, the conjugate operation can be

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replaced by the extended one,

$$\forall u^* \in E^* \quad f^*(u^*) = \sup_{u \in E}\{Re < u^*, u > - f(u)\},$$

where $Re < u^*, u >$ denotes the real part of the complex number $< u^*, u >$.

In what follows, we restrict ourselves to the case of real space since the complex one can be stated in a similar manner.

Let $f \in \widetilde{\mathbb{R}}^E$ and $\lambda > 0$ be real, we define the functionals $\lambda.f$ and $f.\lambda$ by

$$\forall u \in E, \quad (\lambda.f)(u) = \lambda.f(u) \text{ and } (f.\lambda)(u) = \lambda.f(\frac{u}{\lambda}).$$

With this, it is not hard to see that

$$(\lambda.f)^* = f^*.\lambda \text{ and } (f.\lambda)^* = \lambda.f^*.$$

The bi-conjugate of f is the functional $f^{**} : E \longrightarrow \overline{\mathbb{R}}$ defined as follows

$$\forall u \in E \quad f^{**}(u) := (f^*)^*(u) = \sup_{u^* \in E^*}\{< u^*, u > - f^*(u^*)\}.$$

If we denote by $\Gamma_0(E)$ the cone of lower semi-continuous (l.s.c) convex functionals from E into $\widetilde{\mathbb{R}}$ not identically equal to $+\infty$, it is well-known that $f^{**} \leq f$ and, $f^{**} = f$ if and only if $f \in \Gamma_0(E)$. Analogously, we can define $f^{***} : E^* \longrightarrow \overline{\mathbb{R}}$ which satisfies $f^{***} = f^*$, and thus $f^* \in \Gamma_0(E^*)$ for all $f \in \widetilde{\mathbb{R}}^E$.

An important and typical example of $\widetilde{\mathbb{R}}^E$−functional is f_A defined by

$$\forall u \in E \qquad f_A(u) = (1/2) < Au, u >,$$

where $A : E \longrightarrow E^*$ is a bounded linear operator. We say that f_A is quadratic in the sense that $f(t.u) = t^2 f(u)$ for all $u \in E$ and $t \in \mathbb{R}$. It is easy to see that the conjugate operation preserves the quadratic character, and it is well-known that $f_A \in \Gamma_0(E)$ if and only if A is (symmetric) positive. Moreover, if A is positive invertible then f_A^* has the explicit form, [1]

$$\forall u^* \in E^* \qquad f_A^*(u^*) = (1/2) < A^{-1}u^*, u^* > .$$

That is, $f_A^* = f_{A^{-1}}$ and so, as already pointed, the conjugate operation can be considered as a reasonable extension of the inverse operator in the sense of convex analysis. Another important example of $\widetilde{\mathbb{R}}^E$−functional is Ψ_M, namely the indicator functional of the set $M \subset E$, defined by

$$\Psi_M(u) = 0 \text{ if } u \in M, \quad \Psi_M(u) = +\infty \text{ else.}$$

It is known that, $\Psi_M \in \Gamma_0(E)$ if and only if M is a nonempty closed convex of E.

Now, let us recall that, for all $f, g \in \widetilde{\mathbb{R}}^E$ and $\alpha \in]0, 1[$, it is easy to see that

$$(\alpha f + (1-\alpha)g)^* \leq \alpha f^* + (1-\alpha)g^*, \tag{1}$$

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i.e, the map $f \longmapsto f^*$ is convex with respect to the point-wise ordering on $\widetilde{I\!R}^E$.

In the quadratical case: if $f = f_A$ and $g = f_B$ with $A, B : E \longrightarrow E^*$ are positive invertible operators, then the above inequality implies immediately the following one,

$$\forall \alpha \in]0,1[\quad (\alpha.A + (1-\alpha).B)^{-1} \leq \alpha.A^{-1} + (1-\alpha).B^{-1}, \tag{2}$$

for the partial ordering

$$A \leq B \iff B - A \text{ is positive} \iff f_A \leq f_B.$$

It is interesting to notice that, inequality (2) is not obvious to establish directly (i.e without passing from the conjugate operation) as confirmed in [5] for example.

By *dom* f we denote the effective domain of $f \in \widetilde{I\!R}^E$ defined by

$$dom\ f = \{u \in E, \ \ f(u) < +\infty\}.$$

With this, it is clear that $dom\ \Psi_M = M$ and $dom(f+g) = dom\ f \cap dom\ g$.

Finally, for $f, g \in \widetilde{I\!R}^E$ such that $dom\ f \cap dom\ g \neq \emptyset$ and $\lambda \in]0,1[$, the power arithmetic and harmonic functional means of f and g, extending that of operator case, are respectively defined by, [6]

$$\mathcal{A}_\lambda(f,g) = (1-\lambda).f + \lambda.g, \quad \mathcal{H}_\lambda(f,g) = \left((1-\lambda).f^* + \lambda.g^*\right)^*. \tag{3}$$

Clearly, $\mathcal{H}_\lambda(f,g) \in \Gamma_0(E)$ and, if $f, g \in \Gamma_0(E)$ so is $\mathcal{A}_\lambda(f,g)$. Moreover, (1) gives immediately the power arithmetic-harmonic functional mean inequality

$$\forall f, g \in \widetilde{I\!R}^E \quad \mathcal{H}_\lambda(f,g) \ \leq \ \mathcal{A}_\lambda(f,g),$$

which in the quadratic case gives immediately the arithmetic-harmonic operator mean inequality

$$\left((1-\lambda)A^{-1} + \lambda.B^{-1}\right)^{-1} \ \leq \ (1-\lambda)A + \lambda.B,$$

for all positive invertible operators A and B.

Otherwise, it is not hard to verify that, for all $\lambda \in]0,1[$, the map $(f,g) \longmapsto \mathcal{H}_\lambda(f,g)$ defined from $\Gamma_0(E) \times \Gamma_0(E)$ into $\Gamma_0(E)$ is homogenous, monotone increasing, sub-additive and jointly concave. In fact, for $\lambda = 1/2$, these properties are confirmed in [3] and the same arguments work for every $\lambda \in]0,1[$.

Finally, it is very interesting to notice that our convex functionals considered below can take the value $+\infty$. So, the relation $f - f = 0$ is not always true and the two equalities (resp. inequalities) $f = g$ and $f - g = 0$ (resp. $f \leq g$ and $f - g \leq 0$) are not always equivalent.

3. Parameterized Relative Functional Entropy

Let $f, g \in \widetilde{I\!R}^E$ such that $dom\ f \cap dom\ g \neq \emptyset$ and $0 < p < 1$. In [6] we have introduced the power geometric mean (which we called the $(1-p, p)$ functional mean) of f and g by

$$\mathcal{G}_p(f,g) = \frac{\sin\ p\pi}{\pi} \int_0^\infty \frac{t^{p-1}}{1+t} \left(\frac{1}{1+t} f^* + \frac{t}{1+t} g^* \right)^* dt.$$

Setting $s = \frac{t}{1+t}$, it is easy to verify that

$$\mathcal{G}_p(f,g) = \frac{\sin\ p\pi}{\pi} \int_0^1 \frac{t^{p-1}}{(1-t)^p} \mathcal{H}_t(f,g) dt, \tag{4}$$

where

$$\mathcal{H}_t(f,g) = ((1-t).f^* + t.g^*)^*$$

is the power harmonic functional mean.

The properties of $\mathcal{G}_p(f,g)$, which are needed in the sequel, are summarized in the following [6].

Proposition 3.1. *For all $f, g \in \widetilde{I\!R}^E$ such that $dom\ f \cap dom\ g \neq \emptyset$ and $0 < p < 1$, the following statements hold true:*

(i) Conjugate symmetry of $(f,g) \longmapsto \mathcal{G}_p(f,g)$, i.e

$$\mathcal{G}_p(f,g) = \mathcal{G}_{1-p}(g,f).$$

(ii) Power Arithmetic-Geometric-Harmonic Functional Mean inequality, i.e

$$\mathcal{H}_p(f,g) := ((1-p).f^* + p.g^*)^* \leq \mathcal{G}_p(f,g) \leq (1-p).f + p.g := \mathcal{A}_p(f,g).$$

(iii) Extension and conservation for quadratic case, i.e

$$\mathcal{G}_p(f_A, f_B) = f_{G_p(A,B)} \quad \text{with} \;\; G_p(A,B) = A^{1/2}(A^{-1/2}BA^{-1/2})^p A^{1/2}.$$

By virtue of Proposition 3.1, we can prolong the map $]0,1[\ni p \longmapsto \mathcal{G}_p(f,g)$ on $[0,1]$ by setting

$$\mathcal{G}_0(f,g) = f \quad \text{and} \quad \mathcal{G}_1(f,g) = g.$$

Now, we are in a position to state our central definition.

Definition 3.1. Let $f, g \in \widetilde{I\!R}^E$ such that $dom\ f \cap dom\ g \neq \emptyset$ and $p \in (0,1]$. The parameterized relative functional entropy of f and g is defined by

$$\mathcal{R}_p(f/g) := \frac{\mathcal{G}_p(f,g) - f}{p}.$$

Henceforth, when we consider $\mathcal{R}_p(f/g)$ it will be assumed that $dom\ f \cap dom\ g \neq \emptyset$. The first properties of the map $(f,g) \longmapsto \mathcal{R}_p(f/g)$ are discussed in the following results.

Proposition 3.2. *With the above, the following properties are met:*

(i) Invariance translation: for all $f, g \in \widetilde{I\!R}^E$ *and reals* α, β, *we have*

$$\mathcal{R}_p(f+\alpha/g+\beta) = \mathcal{R}_p(f/g) + \beta - \alpha.$$

In particular

$$\mathcal{R}_p(f+\alpha/g+\alpha) = \mathcal{R}_p(f/g).$$

(ii) Functional homogeneity: for all $f, g \in \widetilde{I\!R}^E$ *and* $a, b > 0$ *there hold*

$$\mathcal{R}_p(a.f/b.g) = \sqrt{ab}.\mathcal{R}_p\left(\sqrt{a/b}.f/\sqrt{b/a}.g\right)$$

$$\mathcal{R}_p(f.a/g.b) = \mathcal{R}_p\left(f.\sqrt{a/b}/g.\sqrt{b/a}\right).\sqrt{ab}$$

In particular,

$$\mathcal{R}_p(a.f/a.g) = a.\mathcal{R}_p(f/g) \ \text{ and } \ \mathcal{R}_p(f.a/g.a) = \mathcal{R}_p(f/g).a$$

Proof. Comes from the properties of the conjugate operation recalled in the above section. The details are left to the reader. □

Proposition 3.3. *For all* $f, g \in \widetilde{I\!R}^E$ *we have:*

(i) $\mathcal{R}_p(f/f) = \Psi_{dom\ f}$ *and* $\mathcal{R}_p(f/g) \leq g - f + \Psi_{dom\ f}$.

(ii) $\mathcal{S}(f/g) \leq \int_0^1 \mathcal{R}_t(f/g)dt$.

(iii) $p.\mathcal{R}_p(f/g) - (1-p)\mathcal{R}_{1-p}(g/f) = g - f + \Psi_{dom\ f \cap dom\ g}$.

(iv) $\mathcal{R}_p(f/g) + \mathcal{R}_{1-p}(g/f) \leq 0$ *on* $dom\ f \cap dom\ g$..

Proof. (i) The first equality is immediate and the second one follows from the right inequality of Proposition 3.1(ii), with the fact that $f - f = \Psi_{dom\ f}$.

(ii) It is obvious from the left inequality of Proposition 3.1(ii) when combined with the definition of $\mathcal{S}(f/g)$.

(iii) Use the above points (i) and (ii), with a convenient manipulation.

(iv) It is sufficient to apply the above with Proposition 3.1(i). □

Theorem 3.1. *The map* $(f,g) \longmapsto \mathcal{R}_p(f/g)$ *satisfies the following properties:*

1. *Monotonicity with respect to the second variable: let* $f, g_1, g_2 \in \widetilde{I\!R}^E$ *then*

we have

$$g_1 \le g_2 \implies \mathcal{R}_p(f/g_1) \le \mathcal{R}_p(f/g_2).$$

2. Subadditivity and joint concavity on $\Gamma_0(E)$*: for all* $f_1, f_2, g_1, g_2 \in \Gamma_0(E)$ *and* $\lambda \in]0,1[$ *there hold*

$$\mathcal{R}_p(f_1 + f_2/g_1 + g_2) \ge \mathcal{R}_p(f_1/g_1) + \mathcal{R}_p(f_2/g_2),$$

$$\mathcal{R}_p(\lambda.f_1 + (1-\lambda).f_2/\lambda.g_1 + (1-\lambda).g_2) \ge \lambda.\mathcal{R}_p(f_1/g_1) + (1-\lambda).\mathcal{R}_p(f_2/g_2).$$

Proof. 1. As recalled in the above section, if $g_1 \le g_2$ then $\mathcal{H}_t(f, g_1) \le \mathcal{H}_t(f, g_2)$ for every $t \in]0,1[$, and the desired result follows.

2. Since the map $(f, g) \longmapsto \mathcal{H}_t(f, g)$ is sub-additive and jointly concave for all fixed $t \in]0,1[$ then so is $(f, g) \longmapsto \mathcal{R}_p(f/g)$. This completes the proof. □

The following result shows that the map $(f, g) \longmapsto \mathcal{R}_p(f/g)$ is a reasonable extension of the Tsallis relative entropy $T_p(A/B)$ from positive operators to convex functionals.

Theorem 3.2. *If f and g are quadratic so is $\mathcal{R}_p(f/g)$, with the relationship*

$$\mathcal{R}_p(f_A/f_B) = f_{T_p(A/B)},$$

where $T_p(A/B)$ is the Tsallis relative operator entropy of A and B.

Proof. Follows from Definition 3.1 with Proposition 3.1(iii). □

Combining this latter result with the above ones, we observe that we find immediately some properties (with their proofs) of the map $(A, B) \longmapsto T_p(A/B)$ already stated in the literature and differently proved by many authors (see [4] and the reference cited therein). Precisely, the following result is immediate from the above.

Corollary 3.1. *For all positive invertible linear operators A and B, the map $(A, B) \longmapsto T_p(A/B)$ is homogenous, monotone increasing in B, sub-additive and jointly concave, for the partial ordering:*

$$A \le B \iff B - A \text{ is positive} \iff f_A \le f_B.$$

Now, we are in position to state the next result which tells us that the relative functional entropy, introduced by the author in [7], is the limit when p goes to 0 of the above parameterized family $\mathcal{R}_p(f/g)$. Precisely, we have the following.

Theorem 3.3. *For all $f, g \in \Gamma_0(E)$ such that dom $f \cap$ dom $g \neq \emptyset$, the* *following relationship*

$$\lim_{p\downarrow 0} \mathcal{R}_p(f/g) = \mathcal{S}(f/g)$$

holds true on dom $f \cap$ dom g, where the limit is in the point-wise functional convergence sense.

Proof. First, as announced in [7], it is easy to see that

$$\mathcal{S}(f/g) \leq g - f \text{ on } dom\ f,$$

which, with Proposition 3.3(i) implies that $\mathcal{S}(f/g)$ and $\mathcal{R}_p(f/g)$ are with finite values on *dom* $f \cap$ *dom* g. Now, let $f \in \Gamma_0(H)$. Clearly, $\mathcal{H}_t(f, f) = f$ for all $t \in]0, 1[$ and Proposition 3.1(ii) implies that $\mathcal{G}_p(f, f) = f$ for each $p \in]0, 1[$. This, with relation (4), yields

$$\frac{\sin\ p\pi}{\pi} \int_0^1 \frac{t^{p-1}}{(1-t)^p} dt = 1.$$

Combining this latter formulae with the definition of $\mathcal{R}_p(f/g)$ we can write

$$\mathcal{R}_p(f/g) = \frac{\sin\ p\pi}{p\pi} \int_0^1 \frac{t^{p-1}}{(1-t)^p} \left(\mathcal{H}_t(f, g) - f\right),$$

which, with the fact that

$$\frac{\sin\ p\pi}{p\pi} \frac{t^{p-1}}{(1-t)^p} \left(\mathcal{H}_t(f, g) - f\right) \longrightarrow \frac{\mathcal{H}_t(f, g) - f}{t},$$

when p goes to 0, yields the desired result by a simple application of the theorem on the continuity of a parameter integral. The proof is complete. □

Combining Theorem 3.3 and Theorem 3.2 we obtain immediately the next known result:

$$\lim_{p\downarrow 0} T_p(A/B) = S(A/B),$$

for all positive invertible operators A and B. This also justifies that our approach is a reasonable extension of the relative entropy when the positive linear operator variables are convex functionals.

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