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OPTIMIZATION APPROACH FOR SOLVING SOME GEOMETRY PROBLEMS

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Abstract: Geometry problems of finding inscribed or circumscribed balls defined over a polyhedral set are classical. It is known that finding minimal ellipsoid circumscribing a polytope is NP-hard [5]. Many combinatorial, clustering, data mining, and pattern recognition problems require to find a ball set circumscribing a given set. On the other hand, from computational point of view, it is interesting to find a center of a ball inscribed in a given set defined by a system of linear inequalities. In this paper, we consider a problem of finding the maximum and minimum radiuses of inscribed and circumscribed balls defined over a polyhedral set. We formulate the above problem as optimization problems and then propose some algorithms for solving them.

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1. Introduction

Geometry problems of finding inscribed or circumscribed balls defined over a polyhedral set are classical. It is known that finding minimal ellipsoid circumscribing a polytope is NP-hard [5]. Many combinatorial, clustering, data mining, and pattern recognition problems require to find a ball set circumscribing a given set. On the other hand, from computational point of view, it is interesting to find a center of a ball inscribed in a given set defined by a system

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consider a problem of finding the maximum and minimum radiuses of inscribed and circumscribed balls defined over a polyhedral set. We formulate the above problem as optimization problems and then propose some algorithms for solving them. This paper is organized as follows. In the Section 2, we consider a problem of finding a center of a inscribed ball with the minimum radius. Section 3 is devoted to the minimum radius problem over circumscribed ball.

2. The Maximum Radius of Inscribed Ball

Let us consider a problem of finding the maximum radius of an inscribed ball in a polyhedral set:

$$D = \{x \in \mathbb{R}^n | \ \langle a^i, x\rangle \leq b_i, \ i = 1, 2, \dots, m\}, \quad \mathrm{int}\, D \neq \emptyset,$$

where $a^i \in \mathbb{R}, \ b_i \in \mathbb{R}, \ i = 1, 2, \dots, m$.

Denote by x_{n+1} radius of an inscribed ball in D. Let x be a center of the ball. We can easily show that a ball with radius x_{n+1} and center x belongs to hyper plane $H_i = \{x \in \mathbb{R}^n | \ \langle a^i, x\rangle \leq b_i\}$ if and only if the following conditions hold:

$$\frac{|\langle a^i, x\rangle - b_i|}{||a^i||} \geq x_{n+1}, \quad i = 1, 2, \dots, m.$$

Then our problem is formulated as

$$x_{n+1} \to \max \tag{2.1}$$

$$\text{subject to:} \quad \begin{cases} \frac{|\langle a^i, x\rangle - b_i|}{||a^i||} \geq x_{n+1}\,, \\ \langle a^i, x\rangle \leq b_i, \ i = 1, 2, \dots, m\,, \end{cases} \tag{2.2}$$

which is equivalent to the following linear programming problem

$$x_{n+1} \to \max, \tag{2.3}$$

$$\langle a^i, x\rangle + ||a^i|| x_{n+1} \leq b_i, \quad i = 1, 2, \dots, m. \tag{2.4}$$

Introduce the function $\varphi(x)$ as $\varphi(x) = \min\limits_{1 \leq i \leq m} \frac{|\langle a^i, x\rangle - b_i|}{||a_i||}$, then problem (2.1)-(2.2) can be reduced to the following equivalent problem:

$$\varphi(x) \to \max, \tag{2.5}$$

$$\text{subject to:} \quad \langle a^i, x\rangle \leq b_i, \ i = 1, 2, \dots, m. \tag{2.6}$$

Since function $\varphi(x)$ is nondifferentiable, problem (2.5)-(2.6) can be solved by subgradient methods. In practice, it is easy to implement numerically by solving

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(2.3)-(2.4) as linear programming. The following test problems have been solved numerically.

Problem 1.

$$\begin{cases} x_1 - 8x_2 \le 93\,, \qquad 5x_1 + 7x_2 \ge 146\,, \\ -3x_1 + 8x_2 \le 10\,, \qquad 3x_1 + 6x_2 \le 60\,, \\ 4x_1 + x_2 - 24 \le 0\,. \end{cases}$$

Solution is: $x^* = (10.0223; 9.1379; 3.7110)$, $r^* = 3.711$.

Problem 2.

$$\begin{cases} 3x_1 + 4x_2 \le 50\,, \quad -5x_1 + 12x_2 \le 60\,, \\ 15x_1 - 8x_2 \le 120\,, \quad x_1 \ge 2, \quad x_2 \ge 0\,. \end{cases}$$

Solution is: $x^* = (5.8571, 3.5714, 3.5714)$, $r^* = 3.5714$.

Problem 3.

$$\begin{cases} x_1 + x_2 + x_3 \le 1\,, \quad x_1 - x_2 - x_3 \le 1\,, \\ x_1 - x_2 + x_3 \le 1\,, \quad x_1 + x_2 - x_3 \le 1\,, \\ -x_1 + x_2 + x_3 \le 1\,, \quad -x_1 + x_2 - x_3 \le 0\,. \end{cases}$$

Solution is: $x^* = (-51.6182, -51.8667, 0.2514, 0.2887)$, $r^* = 0.2887$.

Problem 4.

$$\begin{cases} \sum\limits_{i=1}^{100} x_i \le 1\,, \\ x_i \ge 0, \quad i = 1, 2, \dots, 100. \end{cases}$$

The corresponding linear programming problem is formulated as:

$$r = x_{101} \to \max$$

$$\begin{cases} \sum\limits_{i=1}^{100} x_i + 10x_{101} \le 1\,, \\ x_i \ge x_{101}\,, \quad x_i \ge 0, \quad i = 1, 2, \dots, 100\,, \end{cases}$$

and solution is: $x_1^* = x_2^* = \dots = x_{101}^* = 0,0091$, $r^* = x_{101}^*$.

3. Minimum Radius of Circumscribed Ball

Consider a problem of finding the minimum radius of a ball circumscribing the bounded polytope $D = \{x \in \mathbb{R}^n |\ Ax \le b\}$, A -is $(m \times n)$ matrix and $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, b are given. Let y be a center of a ball circumscribing D and x_{n+1} be its radius. Then our problem is formulated as:

$$x_{n+1} \to \min,\ y \in \mathbb{R}^n, \tag{3.1}$$

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$$||y-x|| \le x_{n+1}, \quad \forall x \in D\,, \tag{3.2}$$

which is equivalent to

$$x_{n+1} \to \min,\ y \in \mathbb{R}^n \tag{3.3}$$

$$\text{subject to:} \quad \max_{x\in D} ||y-x|| \le x_{n+1}. \tag{3.4}$$

Now introduce the function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(y) = \max_{x\in D} ||y-x||. \tag{3.5}$$

Lemma 3.1. *Function $f(y)$ is convex on R^n.*

Proof. Take arbitrary points $y^1, y^2 \in \mathbb{R}^n$ and $\alpha \in [0,1]$. Then

$$f(\alpha y^1 + (1-\alpha)y^2) = \max_{x\in D} ||\alpha y^1 + (1-\alpha)y^2 - x||$$
$$= \max_{x\in D} ||\alpha y^1 + (1-\alpha)y^2 - [\alpha x + (1-\alpha)x)]||$$
$$= \max_{x\in D} ||\alpha(y^1-x) + (1-\alpha)(y^2-x)|| \le \max_{x\in D} ||\alpha(y^1-x)|| + \max_{x\in D}(1-\alpha)||y^2-x||$$
$$= \alpha \max_{x\in D} ||y^1-x|| + (1-\alpha)\max_{x\in D} ||y^2-x|| = \alpha f(y^1) + (1-\alpha) f(y^2)$$

and assertion is proved. □

Consider the problem

$$f(y) \to \min,\ y \in \mathbb{R}^n. \tag{3.6}$$

Lemma 3.2. *Problems (3.3)-(3.4) and (3.6) are equivalent.*

Proof. Let x^*_{n+1}, y^* be a solution to problem (3.3)-(3.4). Then

$$\max_{x\in D} ||y^*-x|| \le x^*_{n+1} \le \max_{x\in D} ||y-x|| \le x_{n+1},\ \forall y \in \mathbb{R}^n \ \text{ and } \ \forall x_{n+1} \in \mathbb{R}.$$

Hence, we have $f(y^*) \le f(y)$. In reverse, if $f(y^*) \le f(y)$ then setting $x^*_{n+1} = \max_{x\in D} ||y^*-x|| = f(y^*)$, we arrive at $x^*_{n+1} \le x_{n+1}$ or $\max_{x\in D} ||y^*-x|| \le \max_{x\in D} ||y-x||$. Consequently, (x^*_{n+1}, y^*) is a solution to (3.3)-(3.4). □

Lemma 3.3. *The function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies Lipschitz condition with the constant 1.*

Proof. For arbitrary given $y^1, y^2 \in R^n$, we have

$$f(y^2) - f(y^1) = \max_{x\in D} ||y^2-x|| - \max_{x\in D} ||y^1-x||$$
$$\le \max_{x\in D}[||y^2-x|| - ||y^1-x||] \le \max_{x\in D} ||y^2-y^1|| = ||y^2-y^1||.$$

On the other hand, we can write

$$\begin{aligned} f(y^2) - f(y^1) &= \max_{x\in D} ||y^2 - x|| - \max_{x\in D} ||y^1 - x|| \\ &\geq -\max_{x\in D}[||y^2 - x|| + ||x - y^1||] \geq -\max_{x\in D}[||y^2 - x|| + ||x - y^1||] \\ &\geq -\max_{x\in D} ||y^2 - y^1|| = -||y^2 - y^1||. \end{aligned}$$

Therefore, combining these inequalities, we have

$$|f(y^2) - f(y^1)| \leq ||y^2 - y^1||, \ \forall y^1, y^2 \in \mathbb{R}^n.$$

The proof is complete. □

Corollary. *$f(y)$ is continuous on $\mathbb{R}^n$.*

Lemma 3.4. *The function $f(y)$ defined by* (3.5) *attains its global minimum on $\mathbb{R}^n$.*

Proof. Let $v \in R^n$ be an arbitrary point in $\mathbb{R}^n$. Construct the set L by

$$L(f, v) = \{y \in \mathbb{R}^n | \ f(y) \leq f(v)\}.$$

Since $f(y)$ is continuous, $L(f, v)$ is closed. Denote by K the maximum norm of x over D, i.e., $K = \max_{x\in D} ||x||$. Now we show that $L(f, v)$ is a bounded set. Take an arbitrary $y \in L(f, v)$. Then it is clear that $\max_{x\in D} ||y - x|| \leq f(v)$. Write down the following obvious inequality

$$||y|| = ||y - x + x|| \leq ||y - x|| + ||x||, \ \forall x \in \mathbb{R}^n.$$

Consequently, we have

$$||y|| \leq \max_{x\in D}\{||y - x|| + ||x||\} \leq \max_{x\in D} ||y - x|| + \max_{x\in D} ||x|| \leq f(v) + K.$$

The proof is complete. □

As we can see that finding the minimum radius of a ball circumscribing compact polytope reduces to an unconstrained global minimization problem. But computing $f(y)$ by (3.5) is NP-hard problem and can be solved by an algorithm proposed in [4]. In some cases, for instance, if all vertices of the polytope D are given then the above problem can be solved by numerical methods of nondifferentiable optimization [6].

Let $x^i, \ i = 1, 2, \ldots, m$ be the vertices of the polytope, and x_{n+1} be a radius of a ball circumscribing D. Then the minimum radius problem is formulated as

$$\begin{cases} x_{n+1} \to \min, \ (x, x_{n+1}) \in \mathbb{R}^{n+1}, \\ ||x - x^i|| \leq x_{n+1}, \ i = 1, 2, \ldots, m, \end{cases}$$

which is equivalent to the following minimax problem

$$\begin{cases} \varphi(x) = \max\limits_{1\leq i\leq m} ||x - x^i||^2 \to \min, \ x \in \mathbb{R}^n \,, \\ x_{n+1} = \varphi(x) \,. \end{cases} \tag{3.7}$$

We can show that the function $\varphi(x)$ is convex on $\mathbb{R}^n$. Subdifferential of $\varphi(x)$ is computed as follows [7]:

$$\partial\varphi(x) = \{c \in \mathbb{R}^n | \ c = \sum_{i\in I(x)} \lambda_i(x - x^i), \ \ \sum_{i\in I(x)} \lambda_i = \frac{1}{2}, \ \lambda_i \geq 0\},$$

where $I(x) = \{i| \ f_i(x) = \max\limits_{1\leq i\leq m} f_i(x)\}, \ f_i(x) = ||x - x^i||^2$.

Now we can apply a subgradient method [6] for solving problem (3.4).

Algorithm. *Step 1.* Choose an arbitrary point $x^0 \in \mathbb{R}^n$ and set $k := 0$.

Step 2. If $0 \in \partial\varphi(x^k)$ then x^k is a solution to (3.4).

Step 3. Construct $x^k(\alpha)$ as $x^k(\alpha) = x^k - \alpha c^k, \ \ c^k \in \partial\varphi(x^k), \ \alpha > 0$ and define α_k from condition $\varphi(x^k(\alpha_k)) = \min\limits_{\alpha>0} \varphi(x^k(\alpha))$.

Step 4. Construct $x^{k+1} = x^k(\alpha_k)$.

Step 3. Set $k := k + 1$ and go to Step2.

The convergence of the algorithm is guaranteed by the following statement.

Theorem 3.1. (see [6]) *The sequence* $\{x^k\}, k = 0, 1, \ldots$ *generated by Algorithm is a minimizing sequence for the problem (3.4), that is*

$$\lim_{k\to\infty} \varphi(x^k) = \min_{x\in R^n} \varphi(x).$$

Algorithm was implemented numerically and tested on the following problem.

Problem 1. Given vertices of D are

$$v^1(5, 6), v^2(8, 17), v^3(15, 14), v^4(18, 5), v^5(4, 11).$$

Then problem (3.4) is formulated as:

$$\varphi(x) = \max_{1\leq i\leq 5} \varphi_i(x) \to \min, \ x \in \mathbb{R}^n,$$

with

$$\varphi_1(x_1, x_2) = (x_1 - 5)^2 + (x_2 - 6)^2 \,, \quad \varphi_2(x_1, x_2) = (x_1 - 8)^2 + (x_2 - 17)^2 \,,$$
$$\varphi_3(x_1, x_2) = (x_1 - 15)^2 + (x_2 - 14)^2 \,, \quad \varphi_4(x_1, x_2) = (x_1 - 18)^2 + (x_2 - 5)^2 \,,$$
$$\varphi_5(x_1, x_2) = (x_1 - 4)^2 + (x_2 - 11)^2.$$

A solution found by Algorithm is: $x^* = (11.849; 10.041), \ x^* = \varphi(x^*) = 7.953$.

4. Conclusion

To provide a unified view, we examined some geometry problems defined over a polyhedral set. We reduced the problems to optimization problems and proposed methods and algorithms for solving them. Also, we provide some numerical results.

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