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PULLBACK OF SUBLOCALE

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Abstract: In this paper, we study when the pullback along a morphism of locales preserves spatiality, denseness and give some equivalent characterizations.

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1. Introduction

Recall that a frame A is a complete lattice satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s | s \in S\}$ for all $a \in A$ and $S \subseteq A$. Let A, B be frames, $f : A \to B$ is a frame morphism if f preserves arbitrary joins and finite meets. We write **Frm** for the category of frames and frame morphisms and **Loc** for its dual category whose objects are extensionally the same thing, whose morphisms go in the opposite direction, and write $\mathcal{O}(X)$ for the corresponding frame of a locale X.

A sublocale of a locale X is defined to be a regular subobject of X in **Loc**, i.e., the locale corresponding to a regular quotient of $\mathcal{O}(X)$. We write $\mathrm{Sub}(X)$ for the lattice of sublocales of X, which is dual to the partially ordered set $N\mathcal{O}(X)$ of all nuclei on X under pointwise partial order. We say a sublocale is

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dense if its closure is the whole of X. Every locale has a smallest dense sublocale X_b, defined by setting $\mathcal{O}(X_b) = (\mathcal{O}(X))_{\neg\neg}$. A locale which has enough points is called spatial locale. A locale A is spatial if and only if $A \cong \Omega(X)$ for some space X. For more details of locales please refer to P.T. Johnstone [2], P.T. Johnstone [3].

It is well known that a pullback of a regular monomorphism is a monomorphism in any category. Let $f : Y \to X$ be a continuous map of locales, and j a nucleus on $\mathcal{O}(X)$. Then the pullback of $X_j \rightarrowtail X$ along f is a sublocale of Y, we thus have a map $f^* : N\mathcal{O}(X) \to N\mathcal{O}(Y)$. In [2], P.T. Johnstone has investigated its effect on open and closed nuclei and stated that the pullbacks of open (closed) sublocales along arbitrary morphisms of locale are always open(closed) sublocales. In [4], P.T. Johnstone has shown that a morphism of locales (or toposes) is open if and only if all its pullbacks are skeletal, i.e., pulling back along them preserves denseness of sublocales (or subtoposes). In this paper, we study when the pulling back along a morphism of locales preserves spatiality, denseness and give some equivalent characterizations.

2. Main Results

Throughout this section, we work with the category **Loc**. All objects and morphisms mentioned belong to this category. We refer to epimorphisms and regular monomorphisms in **Loc** as *surjections* and *inclusions* respectively.

We first begin with some lemmas about spatiality.

Lemma 1. *A pullback of a complemented sublocale is complemented, i.e., if $X' \rightarrowtail X$ is a sublocale of X with complement $X'' \rightarrowtail X$ in* **Loc**, *then Y' is complemented in Y, as in*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow i \\ Y & \xrightarrow{f} & X \end{array} \tag{1}$$

Proof. Indeed, finite unions and intersections of sublocales are stable under pullback, then the sublocales $Y' = f^*(X')$ and $Y'' = f^*(X'')$ are complementary obviously in Sub(Y), since $Y' \cup Y'' = f^*(X') \cup f^*(X'') = f^*(X' \cup X'') = Y$, $Y' \cap Y'' = f^*(X') \cap f^*(X'') = f^*(X' \cap X'') = \emptyset$. □

Lemma 2. *Let Y be a spatial locale and X' be a complemented sublocale* *of locale X, then the pullback of X' along f is a spatial sublocale of Y.*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

Proof. Let $X' \rightarrowtail X$ be a sublocale of X with complement $X'' \rightarrowtail X$. So the sublocale $Y' = f^*(X')$ is complemented in Sub(Y) by Lemma 1. And then S' is spatial from the fact that a complemented sublocale of a spatial locale is spatial. This completes the proof. □

It is well known that for a continuous map of locales $f : Y \to X$, the pullback functor f^* has a left adjoint $f_!$:Sub(Y) $\to$ Sub(X) which sends a sublocale S of Y to the image of the composite $S \rightarrowtail Y \to X$. In our paper, the functor $f_!$ plays an important role in proving our main results.

Remark 3. A morphism $f : Y \to X$ in **Loc** is surjective means that the sublocale generated by f is X itself, i.e., $f_*f^*{=}1_X$, which is equivalent to that f^* is a momomorphism.

Lemma 4. *Let $X' \rightarrowtail X$ be a sublocale of X and $f : Y \to X$ a continuous map of locales. Y' be the pullback of X' along f, as in the following square.*

$$\begin{array}{ccc} Y' & \xrightarrow{f_!} & X' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{f} & X \end{array} \tag{2}$$

Then $f_!(Y') = X_{f_*jf^*}$, in particular, if $f_!$ is surjective, then $X' = f_!(Y') = X_{f_*jf^*}$, where f_* is the right adjoint of the corresponding frame morphism of f.

Proof. In above pullback square, the sublocales generated by fj, $if_!$ are the same sublocale, that is, $X_{f_{!*}f_!^*i^*} = X_{f_*jf^*}$, since $fj = if_!$. But, $f_!(Y')$ is a sublocale $X'_{f_{!*}f_!^*}$ of X', its corresponding sublocale of X is $X_{i_*f_{!*}f_!^*i^*}{=}X_{f_{!*}f_!^*i^*}$. So, $f_!(Y') = X_{f_{!*}f_!^*i^*} = X_{f_*jf^*}$.

Thus, if $f_!$ is surjective, by Remark 3 we then have $X' = f_!(Y') = X_{f_{!*}f_!^*i^*} = X_{f_*jf^*}$. □

Lemma 5. *Let $f : Y \to X$ be a locale morphism, then the right adjoint f_* of the corresponding frame morphism preserves prime elements, i.e., $f_*(pt(\mathcal{O}(Y))) \subseteq pt(\mathcal{O}(X))$.*

Proof. Let $a \in pt(\mathcal{O}(Y))$, we want to show $f_*(a) \in pt(\mathcal{O}(X))$. Suppose

$x \wedge y \leq f_*(a)$, then $f^*(x) \wedge f^*(y) = f^*(x \wedge y) \leq f^* f_*(a) \leq a$. So, $f^*(x) \leq a$ or $f^*(y) \leq a$, since a is prime in $\mathcal{O}(Y)$. Thus, $x \leq f_*(a)$ or $y \leq f_*(a)$. □

Proposition 6. *Let $f : Y \to X$ be a locale morphism, the left adjoint $f_!$:Sub$(Y) \to$ Sub(X) of the pullback functor f^* :Sub$(X) \to$ Sub(Y) preserves the spatiality of sublocale, i.e., if Y' is a spatial sublocale of Y, then $f_!(Y')$ is a spatial sublocale of X.*

Proof. As in diagram (2), suppose $a \in f_!(Y')$, then $a = f_* j f^*(a)$, $j f^*(a) \in Y'$. So $j f^*(a)$ can be expressed as a meet of primes in Y', since Y' is spatial, i.e.,

$$jf^*(a) = \bigwedge_{Y'} \{p \in pt(Y') | jf^*(a) \leq p\} .$$

Thus,

$$a = f_* j f^*(a) = f_*(\bigwedge_{Y'} \{p \in pt(Y') | jf^*(a) \leq p\}) = \bigwedge_{f_!(Y')} \{f_*(p) \in pt(f_!(Y')) | jf^*(a) \leq p\} .$$

By Lemma 5, we have $f_*(pt(Y')) \subseteq pt(f_!(Y'))$. So,

$$a = \bigwedge_{f_!(Y')} \{f_*(p) \in pt(f_!(Y')) | a \leq f_*(p)\} .$$

This means a can be expressed as a meet of primes in $f_!(Y')$. Whence, X' is spatial. □

Proposition 7. *Let $f : Y \to X$ be a locale morphism, the left adjoint $f_!$:Sub$(Y) \to$ Sub(X) of the pullback functor f^* :Sub$(X) \to$ Sub(Y) preserves the denseness of sublocale, i.e., if Y' is a dense sublocale of Y, then $f_!(Y')$ is a dense sublocale of $f_!(Y)$. In particular, if f is surjective, then $f_!(Y')$ is a dense sublocale of X.*

Proof. As in the diagram (2). Since Y' is dense in Y, so $j(0_Y) = 0_Y$. And it follows easily from Lemma 4 that $f_!(Y') = X_{f_* j f^*}$, so $f_* j f^*(0_X) = f_* j(0_Y) = f_*(0_Y) = f_* f^*(0_X)$, it shows that $f_!(Y')$ is a dense sublocale of $f_!(Y)$.

If f is surjective, then f^* is monomorphism and then f_* is surjective, so $f_* f^*(0_X) = 0_X$, thus $f_!(Y')$ is dense in X. □

To get our main result about spatiality, we need also the following lemma.

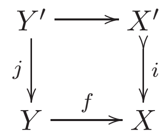
Lemma 8. (see P.T. Johnstone [4]) *Arbitrary surjections are stable under pullback along complemented inclusions in* **Loc**.

Proof. Let $f : Y \to X$ be a surjection, and let $X' \to X$ be a sublocale of X with complement $X'' \rightarrowtail X$, the sublocales $Y' = f^*(X')$ and $Y'' = f^*(X'')$

are complementary in $Sub(Y)$; in particular $Y' \cup Y'' = Y$, so since $f_!$ preserves unions we have $f_!(Y') \cup f_!(Y'') = X$. But $f_!(Y') \leq X'$ and $f_!(Y'') \leq X''$; since $X' \cap X'' = \emptyset$, so $f_!(Y') = X'$, i.e., the pullback of f is a surjection $Y' \to X'$. □

We are now in a position to state the main result.

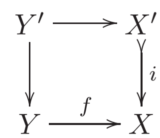
Theorem 9. *Let X, Y be spatial locales and $f : Y \to X$ be a surjection. X' is complemented in Sub(X), Y' is the pullback of X' along morphism $f : Y \to X$, as the following square shows. Then X' is spatial iff Y' is spatial*



Proof. (⇒): By Lemma 2, it can be concluded.

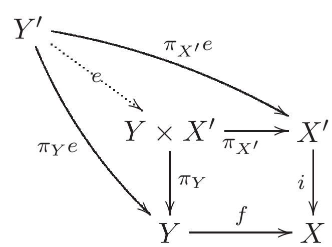
(⇐): By Lemma 4 and Lemma 8, we have $f_!(Y') = X'$. By Proposition 6, then X' is spatial. □

Proposition 10. *For a pullback square in* **Loc**,



Let X be a regular locale and Y be a spatial locale. X' is complemented in Sub(X), then Y' is spatial whenever X' is locally compact.

Proof. The pullback Y' can be considered as the equalizer of parallel morphisms $f\pi_Y, i\pi_{X'} : Y \times X' \to X$. Where $\pi_Y : Y \times X' \to Y, \pi_{X'} : Y \times X' \to X'$ are projections



So Y' is a closed sublocale of $Y \times X'$, since X is regular. If X' is locally compact, then $Y \times X'$ is spatial. Thus Y' is a spatial locale from the fact that a closed sublocale of a spatial locale is spatial. □

We now turn to discuss the denseness-preserving.

A locale map $f : X \to Y$ is said to be *skeletal* if $\neg f^*(\neg U) = \neg\neg f^*(U)$

for all $U \in \mathcal{O}(X)$. And for a locale map $f : Y \to X$, the pullback of any dense sublocale of X along f is a dense sublocale of Y iff f is skeletal (see P.T. Johnstone [4]).

Similar to the discussion about spatiality, we give the following theorem.

Theorem 11. *For a pullback square in* **Loc**,

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ {\scriptstyle j^{op}}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{f} & X \end{array}$$

X' is complemented in Sub(X), Y' is dense, then X' is dense whenever $f : Y \to X$ is surjective. If f is skeletal also, then X' is a dense iff Y' is a dense.

Proof. By Lemma 4 and Lemma 8, we have $f_!(Y') = X'$. Then the proof is just a matter of showing the denseness of $f_!(Y')$. By Proposition 7, it follows that X' is dense in X. The left is obvious by the definition of skeletal morphism. □

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