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ON EINSTEIN LORENTZIAN α-SASAKIAN MANIFOLDS

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Abstract: In the present paper we study an Einstein Lorentzian α-Sasakian manifolds. Here we have shown that an Einstein Lorentzian α-Sasakian manifold satisfying $R(X,Y)P = 0$ and $R(X,Y)N = 0$, where P is projective curvature tensor and N is conharmonic curvature tensor and is locally isometric to a unit sphere $S^n(\alpha)$.

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1. Introduction

In [14], S. Tanno classified connected almost contact metric manifolds whose automorphism group possesses the maximum dimension. For such a manifold, the sectional curvature of a plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes:

(1) homogeneous normal contact Riemannian manifolds with $c > 0$,

(2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and

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(3) a warped product space $\mathbf{R} \times_f \mathbf{C}$ if $c > 0$.

It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu [8] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [8]. In the Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [6]. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [13] if the product manifold $M \times \mathbf{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ (see [11], [12]) coincides with the class of the trans-Sasakian structures of type (α, β). In fact, in [12], local nature of the two subclasses, namely, C_5 and C_6 structures, of trans-Sasakian structures are characterized completely. We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [3], β-Kenmotsu [8] and α-Sasakian [8] respectively.

In 2005, Ahmet Yildiz [16] studied Lorentzian α-Sasakian manifolds and proved that conformally flat and quasi-coformally flat Lorentzian α-Sasakian manifolds are locally isometric with a sphere S^{2^n+1}. And in 2007, C.S. Bagewadi and Venkatesha [1] studied trans-Sasakian manifolds satisfying $R(X, Y).C = 0$, $R(X, Y).P = 0$ and $R(X, Y).\bar{P} = 0$ and they obtained some useful results.

In this paper we study Einstein Lorentzian α-Sasakian manifold satisying $R(X, Y).P = 0$ and $R(X, Y).N = 0$, where P is projective curvature tensor and N is conharmonic curvature tensor respectively, and show that such a manifold is of constant scalar curvature in both the cases.

2. Preliminaries

A differentiable manifold M of dimension n is called Lorentzian α-Sasakian manifold if it admits a $(1, 1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy (see [16], [4])

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.5}$$

for all $X, Y \in TM$.

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Also Lorentzian α-Sasakian manifold M is satisfying (see [5])

$$(a)\ \nabla_X \xi = -\alpha\phi X, \quad (b)\ (\nabla_X \eta)(Y) = -\alpha g(\phi X, Y), \tag{2.6}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Further, on Lorentzian α-Sasakian manifold M the following relations hold (see [15], [10])

$$\eta(R(X,Y)Z) = \alpha^2(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)), \tag{2.7}$$
$$R(\xi,X)Y = \alpha^2(g(X,Y)\xi - \eta(Y)X), \tag{2.8}$$
$$R(X,Y)\xi = \alpha^2(\eta(Y)X - \eta(X)Y), \tag{2.9}$$
$$R(\xi,X)\xi = \alpha^2(\eta(X)\xi + X), \tag{2.10}$$
$$(\nabla_X\phi)(Y) = \alpha[g(X,Y)\xi - \eta(Y)X)], \tag{2.11}$$
$$S(X,\xi) = (n-1)\alpha^2\eta(X), \tag{2.12}$$
$$g(R(\xi,X)Y,\xi) = -\alpha^2[g(X,Y) + \eta(X)\eta(Y)], \tag{2.13}$$

The projective curvature tensor P [1] and conharmonic curvature tensor N [9] on a Riemannian manifold M are defined respecively as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\left[S(Y,Z)X - S(X,Z)Y\right], \tag{2.14}$$

$$N(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)r(X) - g(X,Z)r(Y)\right], \tag{2.15}$$

where R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature.

3. An Einstein Lorentzian α-Sasakian Manifold Satisfying $R(X,Y).P = 0$

Let the Riemannian manifold M be Einstein manifold, then (2.14) gives

$$P(X,Y)Z = R(X,Y)Z - \frac{k}{n-1}\left[g(Y,Z)X - g(X,Z)Y\right]. \tag{3.1}$$

Now,

$$\begin{aligned}\eta(P(X,Y)Z) &= g(P(X,Y)Z,\xi)\\ &= g(R(X,Y)Z - \frac{k}{n-1}[g(Y,Z)X - g(X,Z)Y],\xi)\\ &= \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\end{aligned}$$

$$- \frac{k}{n-1}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]$$

or

$$\eta(P(X,Y)Z) = \left(\alpha^2 - \frac{k}{n-1}\right)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\,. \tag{3.2}$$

Taking $X = \xi$ in (3.2) and using (2.1) and (2.4) we get

$$\eta(P(\xi,Y)Z) = \left(\frac{k}{n-1} - \alpha^2\right)[g(Y,Z) + \eta(Y)\eta(Z)]\,. \tag{3.3}$$

Again, taking $Z = \xi$ in (3.2) and using (2.1) and (2.4), we get

$$\eta(P(X,Y)\xi) = 0. \tag{3.4}$$

Now,

$$\begin{aligned}(R(X,Y)P)(U,V)W = R(X,Y)P(U,V)W - P(R(X,Y)U,V)W \\ -P(U,R(X,Y)V)W - P(U,V)R(X,Y)W\,.\end{aligned} \tag{3.5}$$

Let $R(X,Y).P = 0$. Then we have

$$\begin{aligned}R(X,Y)P(U,V)W - P(R(X,Y)U,V)W \\ -P(U,R(X,Y)V)W - P(U,V)R(X,Y)W = 0\,.\end{aligned} \tag{3.6}$$

Therefore,

$$\begin{aligned}g[R(X,Y)P(U,V)W,\xi] - g[P(R(X,Y)U,V)W,\xi] \\ -g[P(U,R(X,Y)V)W,\xi] - g[P(U,V)R(X,Y)W,\xi] = 0.\end{aligned} \tag{3.7}$$

Put $X = \xi$ in (3.10) we get

$$\begin{aligned}g[R(\xi,Y)P(U,V)W,\xi] - g[P(R(\xi,Y)U,V)W,\xi] \\ -g[P(U,R(\xi,Y)V)W,\xi] - g[P(U,V)R(\xi,Y)W,\xi] = 0.\end{aligned} \tag{3.8}$$

From this it follows that

$$\begin{aligned}-\alpha^2\tilde{P}(U,V,W,Y) - \eta(Y)\eta(P(U,V)W) + \eta(U)\eta(P(Y,V)W) \\ +\eta(V)\eta(P(U,Y)W) + \eta(W)\eta(P(U,V)Y) - \alpha^2 g(Y,U)\eta(P(\xi,V)W). \\ -\alpha^2 g(Y,V)\eta(P(U,\xi)W) = 0,\end{aligned} \tag{3.9}$$

where

$$\tilde{P}(U,V,W,Y) = g(P(U,V)W,Y)\,.$$

Taking $Y = U$ in (3.9), we get

$$\begin{aligned}\tilde{P}(U,V,W,U) + \eta(V)\eta(P(U,U)W) + \eta(W)\eta(P(U,V)U) \\ -g(U,U)\eta(P(\xi,V)W) - g(U,V)\eta(P(U,\xi)W) = 0.\end{aligned} \tag{3.10}$$

Let $\{e_i\}$ $i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \le i \le n$ of the relation (3.10) for $U = e_i$, gives

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$$\eta(P(\xi,V)W) = -\frac{1}{n}S(V,W) + \left[\frac{k}{n} + \frac{k}{n(n-1)} - \frac{\alpha^2}{n}\right] g(V,W) + \left[\frac{k}{n} - \frac{(n-1)\alpha^2}{n} + \frac{k}{n(n-1)} - \frac{\alpha^2}{n}\right] \eta(V)\eta(W)\,. \tag{3.11}$$

From (3.3) and (3.14)

$$S(V,W) = (n-1)\alpha^2 g(V,W)\,. \tag{3.12}$$

This gives

$$k = (n-1)\alpha^2\,. \tag{3.13}$$

Using (3.13) in (3.9), we get

$$-\tilde{P}(U,V,W,Y) = 0. \tag{3.14}$$

From (3.14) it follows that

$$P(U,V)W = 0. \tag{3.15}$$

Therefore the Einstein Lorentzian α-Sasakian manifold is projectively flat. Hence we can state

Theorem 3.1. *If in an Einstein Lorentzian α-Sasakian manifold M, the relation $R(X,Y).P = 0$ holds, then the manifold is projectively flat.*

4. Projectively flat Einstein Lorentzian α-Sasakian Manifold

In this section we suppose that $P(X,Y)Z = 0$. Then from (2.14) we get

$$R(X,Y)Z = \frac{k}{n-1}[g(Y,Z)X - g(X,Z)Y]. \tag{4.1}$$

From (4.1), we get

$$\tilde{R}(X,Y,Z,W) = \frac{k}{n-1}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{4.2}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y,Z)W)$.

Putting $X = W = \xi$ in (4.2) and using (2.1), (2.4) and (2.13), we get

$$\left(\alpha^2 - \frac{k}{n-1}\right)[g(Y,Z) + \eta(Y)\eta(Z)] = 0\,. \tag{4.3}$$

This shows that either $k = \alpha^2(n-1)$ or $g(Y,Z) = -\eta(Y)\eta(Z)$. But if $g(Y,Z) = -\eta(Y)\eta(Z)$, then from (2.3) we get $g(\phi Y, \phi Z) = 0$, which is not possible. Therefore, $k = \alpha^2(n-1)$. Now putting $k = \alpha^2(n-1)$ in (4.1), we get

$$R(X,Y)Z = \alpha^2[g(Y,Z) + \eta(Y)\eta(Z)]\,. \tag{4.4}$$

Therefore the manifold is of constant scalar curvature α^2. Hence we can state

Theorem 4.2. *A projectively flat Einstein Lorentzian α-Sasakian manifold is locally isometric to a sphere $S^n(c)$, where $c = \alpha^2$.*

5. An Einstein Lorentzian α-Sasakian Manifold Satisfying $R(X,Y).N = 0$

Let the Riemannian manifold M be Einstein manifold, then (2.15) simplifies to

$$N(X,Y)Z = R(X,Y)Z - \frac{2k}{n-2}[g(Y,Z)X - g(X,Z)Y]. \tag{5.1}$$

Now

$$\eta(N(X,Y)Z) = \left(\alpha^2 - \frac{2k}{n-2}\right)[g(Y,Z)X - g(X,Z)Y]. \tag{5.2}$$

Taking $X = \xi$ in (5.2) and using (2.1) and (2.4) we get

$$\eta(N(\xi,Y)Z) = \left(\frac{2k}{n-2} - \alpha^2\right)[g(Y,Z)X + \eta(Y)\eta(Z)]. \tag{5.3}$$

Again taking $Z = \xi$ in (5.2) and then using (2.1) and (2.4) we get

$$\eta(N(X,Y)Z) = 0. \tag{5.4}$$

Now,

$$\begin{aligned}(R(X,Y)N)(U,V)W &= R(X,Y)N(U,V)W - N(R(X,Y)U,V)W \\ &\quad - N(U,R(X,Y)V)W - N(U,V)R(X,Y)W\,.\end{aligned}$$

Let $R(X,Y).N = 0$. Then we have,

$$\begin{aligned}&R(X,Y)N(U,V)W - N(R(X,Y)U,V)W \\ &- N(U,R(X,Y)V)W - N(U,V)R(X,Y)W = 0.\end{aligned} \tag{5.5}$$

Put $X = \xi$ in (5.5)

$$\begin{aligned}&g[R(\xi,Y)N(U,V)W,\xi] - g[N(R(\xi,Y)U,V)W,\xi] \\ &- g[N(U,R(\xi,Y)V)W,\xi] - g[N(U,V)R(\xi,Y)W,\xi] = 0.\end{aligned}$$

From this it follows that

$$\begin{aligned}&\alpha^2[-\tilde{N}(U,V,W,Y) - \eta(Y)\eta(N(U,V)W) + \eta(U)\eta(N(Y,V)W) \\ &+ \eta(V)\eta(N(U,Y)W) + \eta(W)\eta(N(U,V)Y) - g(Y,U)\eta(N(\xi,V)W) \\ &\qquad - g(Y,V)\eta(N(U,\xi,)W)] = 0,\end{aligned} \tag{5.6}$$

where

$$\tilde{N}(U,V,W,Y) = g(N(U,V)W,Y).$$

Taking $Y = U$, in (5.6), we have

$$-\tilde{N}(U,V,W,Y) - \eta(Y)\eta(N(U,V)W) + \eta(U)\eta(N(Y,V)W) + \eta(V)\eta(N(U,Y)W) + \eta(W)\eta(N(U,V)Y) - g(Y,U)\eta(N(\xi,V)W) - g(Y,V)\eta(N(U,\xi,)W) = 0.$$

Let $\{e_i\}$ $i = 1, 2, ..., n$ be an otrhogonal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (5.7) for $U = e_i$, gives

$$\eta(N(\xi,V)W) = -\frac{1}{n}S(V,W) + \frac{1}{n}\left[\frac{2nk}{n-2} - \alpha^2\right]g(V,W) + \frac{1}{n}\left[\frac{2(n-1)k}{n-2} - n\alpha^2 + \frac{2k}{n-2}\right]\eta(V)\eta(W)\,. \tag{5.7}$$

From (5.3) and (5.7) we get

$$S(V,W) = \frac{2(n-1)k}{n-2}g(V,W) + (n-1)\left[\frac{2k}{n-2} - \alpha^2\right]\eta(V)\eta(W) \tag{5.8}$$

Taking $W = \xi$ in (5.8) and simplifying using (2.12), we obtain

$$k = \frac{(n-2)\alpha^2}{2}. \tag{5.9}$$

Using (5.9) in (5.6), we get

$$-\tilde{N}(U,V,W,Y) = 0. \tag{5.10}$$

From (5.10) it follows that

$$N(U,V)W = 0. \tag{5.11}$$

Therefore the Einstein Lorentzian α-Sasakian manifold is conharmonically flat. Hence we can state

Theorem 5.3. *If in an Einstein Lorentzian α-Sasakian manifold M, the relation $R(X,Y).N = 0$ holds, then the manifold is conharmonically flat.*

6. Conharmonically flat Einstein Lorentzian α-Sasakian Manifold

In this section we suppose that $N(X,Y)Z = 0$. Then from (2.15) we get

$$R(X,Y)Z = \frac{2k}{n-2}[g(Y,Z)X - g(X,Z)Y]. \tag{6.1}$$

From (6.1), we get

$$\tilde{R}(X,Y,Z,W) = \frac{2k}{n-2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{6.2}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y,Z)W)$.

Putting $X = W = \xi$ in (6.2) and using (2.1), (2.4) and (2.13), we get

$$\left(\alpha^2 - \frac{2k}{n-2}\right)[g(Y,Z) + \eta(Y)\eta(Z)] = 0. \tag{6.3}$$

This shows that either $k = \frac{(n-2)\alpha^2}{2}$ or $g(Y,Z) = -\eta(Y)\eta(Z)$. But if $g(Y,Z) = -\eta(Y)\eta(Z)$, then from (2.3), we get $g(\phi Y, \phi Z) = 0$, which is not possible. Therefore, $k = \frac{(n-2)\alpha^2}{2}$. Now putting $k = \frac{(n-2)\alpha^2}{2}$ in (6.1), we get

$$R(X,Y)Z = \alpha^2[g(Y,Z) + \eta(Y)\eta(Z)]\,. \tag{6.4}$$

Therefore the manifold is of constant scalar curvature α^2. Hence we can state

Theorem 6.4. *A conharmonically flat Einstein Lorentzian α-Sasakian manifold is locally isometric to a sphere $S^n(c)$, where $c = \alpha^2$.*

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