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GENERALISED GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS

Sanjib Kumar Datta[§], Sourav Kar[2]

[1]Department of Mathematics
University of North Bengal
Raja Rammohunpur, District Darjeeling
PIN Code 734013, West Bengal, INDIA
e-mail: sanjib_kr_datta@yahoo.co.in

[2]Department of Mathematics
Siliguri Institute of Technology
Sukna, District Darjeeling
PIN Code 734009, West Bengal, INDIA
e-mail: sourav_kar_nbu@yahoo.co.in

Abstract: The aim of the paper is to study the comparative growth properties of composite entire functions using L^*-order and L^*-type where $L = L(r)$ is a slowly changing function. Further we introduce the notion of generalised relative L^*-order of entire functions and discuss some of their properties.

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1. Introduction, Definitions and Notations

Let $L = L(r)$ be a positive continuous function increasing slowly, i.e. $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Singh and Barker defined it in the following way:

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[§]Correspondence address: 25, School Road, Kalianibash, Barrackpore, P.O.: Nonachandanpukur, Dist.: 24 Parganas (North), P.S.: Titagarh, PIN CODE: 743102, West Bengal, INDIA

Definition 1. (see Singh and Barker [4]) A positive continuous function $L(r)$ is called a 'slowly changing function' if for $\varepsilon > 0$,

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon}$$

for $r \geq r(\varepsilon)$ and uniformly for $k(\geq 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0.$$

Somasundaram and Thamizharasi [5] introduced the notions of L-order and L-type for entire functions defined in the open complex plane $\mathbb{C}$. The more generalised concept for L-order and L-type of entire functions are L^*-order and L^*-type respectively. Their definitions are as follows:

Definition 2. (see Somasundaram and Thamizharasi [5]) The L^*-order $\rho_f^{L^*}$ and L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}\,,$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Definition 3. (see Somasundaram and Thamizharasi [5]) The L^*-type $\sigma_f^{L^*}$ of an entire function f is defined as follows

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}},\ 0 < \rho_f^{L^*} < \infty.$$

In the line of Definition 2 and Definition 3 we may introduce the following two definitions.

Definition 4. The t-th generalised L^*-order ${}^{(t)}\rho_f^{L^*}$ and t-th L^*-lower order ${}^{(t)}\lambda_f^{L^*}$ of an entire function f are defined in the following way

$${}^{(t)}\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log[re^{L(r)}]} \quad \text{and} \quad {}^{(t)}\lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log[re^{L(r)}]}\,,$$

where $t = 2, 3, ...$.

Definition 5. The t-th generalised L^*-type ${}^{(t)}\sigma_f^{L^*}$ of an entire function f is defined as

$${}^{(t)}\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[t-1]} M(r,f)}{[re^{L(r)}]^{{}^{(t)}\rho_f^{L^*}}},\ \text{for } t = 2, 3, ... \text{ and } 0 <{}^{(t)} \rho_f^{L^*} < \infty.$$

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Lakshminarasimhan [2] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [3] worked on the entire functions of L-bounded index and of non uniform L-bounded index. In the paper we intend to establish some results on the comparative growth properties of composite entire functions using generalised L^*-order and L^*-type.

Let f and g be two entire functions and $F(r) \equiv M(r,f) = \max\{|f(z)| : |z| = r\}$, $G(r) \equiv M(r,g) = \max\{|g(z)| : |z| = r\}$. If f is non constant then $F(r)$ is strictly increasing and continuous and its inverse $F^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that

$$\lim_{s\to\infty} F^{-1}(s) = \infty.$$

Bernal [1] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$, as follows:

$$\rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\} = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

Similarly one may define the relative lower order of f with respect to g, denoted by $\lambda_g(f)$ in the following manner

$$\lambda_g(f) = \sup\{\mu' > 0 : F(r) > G(r^{\mu'}) \text{ for all } r > r_0(\mu') > 0\} = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

The above two definitions coincide with the classical definitions of order and lower order if $g(z) = \exp z$ (Titchmarsh [6]).

In the paper we introduce the definition of generelised relative L^*-order and generelised relative L^*-lower order of f with respect to g where f and g are both entire functions and study some of their properties.

Definition 6. If $t \geq 1$ is a positive integer, then the t-th generalised relative L^*-order and t-th generalised relative L^*-lower order of an entire function f with respect to an entire function g, denoted respectively by ${}^{(t)}\rho_g^{L^*}(f)$ and ${}^{(t)}\lambda_g^{L^*}(f)$ are defined by

$${}^{(t)}\rho_g^{L^*}(f) = \inf\{\mu_0 > 0 : F(r) < G(\exp^{[t-1]}[re^{L(r)}]^{\mu_0}) \text{ for all } r > r_0(\mu_0) > 0\} = \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(r)}{\log[re^{L(r)}]},$$

where $\exp^{[t]} x = \exp(\exp^{[t-1]} x)$ and $\exp^{[0]} x = x$ and

$${}^{(t)}\lambda_g^{L^*}(f) = \sup\{\mu_0' > 0 : F(r) > G(\exp^{[t-1]}[re^{L(r)}]^{\mu_0'}) \text{ for all } r > r_0(\mu_0') > 0\}$$

$$= \liminf_{r\to\infty} \frac{\log^{[t]} G^{-1}F(r)}{\log[re^{L(r)}]} \text{ for } t = 1, 2, 3, \ldots .$$

Definition 7. (Bernal [1]) Two entire functions f and g are said to be asymtotically equivalent if there exists l, $0 < l < \infty$ such that

$$\frac{F(r)}{G(r)} \to l \quad \text{as } r \to \infty,$$

and in this case we write $f \sim g$.

If $f \sim g$, then clearly $g \sim f$.

We do not explain the standard definitions and notations in the theory of entire functions as those are available in [7]. Throughout the paper we shall assume f, g, h, etc. to be non-constant and if they are entire then $F(r)$, $G(r)$, $H(r)$, etc. denote respectively their maximum modulus on $|z| = r$.

2. Lemmas

In this section we present some lemmas which will be needed in sequel.

Lemma 1. (see Bernal [1]) *Let f be entire and $\alpha > 1$, $0 < \beta < \infty$. Then $F(\alpha r) > \beta F(r)$ for all large r.*

Lemma 2. (see Bernal [1]) *Let f be an entire function. If $\alpha > 1$, $0 < \beta < \alpha$, $s > 1$, $0 < \mu < \lambda$ and n is a positive integer, then:*

(a) $F(\alpha r) > \beta F(r)$.

(b) $\lim_{r\to\infty} \frac{F(r^s)}{r^n F(r)} = \infty = \lim_{r\to\infty} \frac{F(r^\lambda)}{r^n F(r^\mu)}$, *for transcendental f.*

Lemma 3. *Let P be a polynomial and f be a transcendental entire function. Then for any entire function g, ${}^{(t)}\rho_g^{L^*}(Pf) = {}^{(t)}\rho_g^{L^*}(f)$ where $t = 1, 2, 3, \ldots$.*

Proof. Let $P(z)$ be a polynomial of degree m. Then we can always choose a positive number α, $0 < \alpha < 1$ and choose a positive integer $n(> m)$ such that

$$2\alpha < |P(z)| < r^n \tag{*}$$

hold on $|z| = r$ for all large r.

So by the first part of Lemma 2 we get that

$$F(r) = F(\frac{1}{\alpha}.\alpha r) > \frac{1}{2\alpha} F(\alpha r)\,, \quad \text{i.e.} \quad F(\alpha r) < 2\alpha F(r).$$

Let $h(z) = P(z)f(z)$. Then by the second part of Lemma 2 and (*) it follows for $s > 1$ that

$$F(\alpha r) < 2\alpha F(r) < H(r) < r^n F(r) < F(r^s), \quad \text{i.e.} \quad F(\alpha r) < H(r) < F(r^s).$$

This gives that

$$\limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(\alpha r)}{\log[\alpha r e^{L(\alpha r)}]} . \frac{\log[\alpha r e^{L(\alpha r)}]}{\log[r e^{L(r)}]} \le \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}H(r)}{\log[r e^{L(r)}]}$$

$$\le \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(r^s)}{\log[r^s e^{L(r^s)}]} . \frac{\log[r^s e^{L(r^s)}]}{\log[r e^{L(r)}]},$$

$$\text{i.e.} \quad {}^{(t)}\rho_g^{L^*}(f) \le^{(t)} \rho_g^{L^*}(h) \le s.{}^{(t)}\rho_g^{L^*}(f).$$

Since $s > 1$ is arbitrary, letting $s \to 1 + 0$, we obtain that

$${}^{(t)}\rho_g^{L^*}(Pf) =^{(t)} \rho_g^{L^*}(f).$$

This proves the lemma. □

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f and g be two entire functions such that $0 <^{(t)} \lambda_{fog}^{L^*} \le {}^{(t)}\rho_{fog}^{L^*} < \infty$ and $0 <^{(t)} \lambda_g^{L^*} \le^{(t)} \rho_g^{L^*} < \infty$. Then for any positive number A,*

$$\frac{{}^{(t)}\lambda_{fog}^{L^*}}{A^{(t)}\rho_g^{L^*}} \le \liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \le \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A^{(t)}\lambda_g^{L^*}}$$

$$\le \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, L(g))} \le \frac{{}^{(t)}\rho_{fog}^{L^*}}{A^{(t)}\lambda_g^{L^*}},$$

where $t = 2, 3, ...$

Proof. From the definition of t-th generalised L^*-order and t-th generalised L^*-lower order we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[t]} M(r, fog) \ge ({}^{(t)}\lambda_{fog}^{L^*} - \varepsilon) \log[r e^{L(r)}] \tag{1}$$

and

$$\log^{[t]} M(r^A, g) \le A({}^{(t)}\rho_g^{L^*} + \varepsilon) \log[r e^{L(r)}]. \tag{2}$$

Now from (1) and (2) it follows for all sufficiently large values of r,

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{({}^{(t)}\lambda_{fog}^{L^*} - \varepsilon)}{A({}^{(t)}\rho_g^{L^*} + \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A{}^{(t)}\rho_g^{L^*}}. \tag{3}$$

Again for a sequence of values of r tending to infinity

$$\log^{[t]} M(r, fog) \leq ({}^{(t)}\lambda_{fog}^{L^*} + \varepsilon) \log[re^{L(r)}] \tag{4}$$

and for all large values of r,

$$\log^{[t]} M(r^A, g) \geq A({}^{(t)}\lambda_g^{L^*} - \varepsilon) \log[re^{L(r)}]. \tag{5}$$

So combining (4) and (5) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\lambda_{fog}^{L^*} + \varepsilon}{A({}^{(t)}\lambda_g^{L^*} - \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that,

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A{}^{(t)}\lambda_g^{L^*}}. \tag{6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r^A, g) \leq A({}^{(t)}\lambda_g^{L^*} + \varepsilon) \log[re^{L(r)}]. \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\lambda_{fog}^{L^*} - \varepsilon}{A({}^{(t)}\lambda_g^{L^*} + \varepsilon)}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A{}^{(t)}\lambda_g^{L^*}}. \tag{8}$$

Also for all large values of r,

$$\log^{[t]} M(r, fog) \leq ({}^{(t)}\rho_{fog}^{L^*} + \varepsilon) \log[re^{L(r)}]. \tag{9}$$

So from (5) and (9) it follows for all sufficiently large values of r,

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*} + \varepsilon}{A({}^{(t)}\lambda_g^{L^*} - \varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A^{(t)}\lambda_g^{L^*}}. \tag{10}$$

Thus the theorem follows from (3), (6), (8) and (10). □

Remark 1. The sign '$\leq$' cannnot be replaced by '$<$' only in Theorem 1 as we see in the following example.

Example 1. Let $f = z$, $g = \exp z$, $A = 1$, $t = 2$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any real number. So

$${}^{(2)}\rho_{fog}^{L^*} = {}^{(2)}\lambda_{fog}^{L^*} = {}^{(2)}\rho_g^{L^*} = {}^{(2)}\lambda_g^{L^*} = 1.$$

Also

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} = 1 = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)}.$$

Theorem 2. *Let f and g be two entire functions such that $0 < {}^{(t)}\lambda_{fog}^{L^*} \leq {}^{(t)}\rho_{fog}^{L^*} < \infty$ and $0 < {}^{(t)}\rho_g^{L^*} < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A^{(t)}\rho_g^{L^*}} \leq \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)},$$

where $t = 2, 3, \ldots$

Proof. From the definition of t-th generalised L^*-order we get for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r^A, g) \geq A({}^{(t)}\rho_g^{L^*} - \varepsilon)\log[re^{L(r)}]. \tag{11}$$

Now from (9) and (11) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*} + \varepsilon}{A({}^{(t)}\rho_g^{L^*} - \varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A^{(t)}\rho_g^{L^*}}. \tag{12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r, fog) \geq ({}^{(t)}\rho_{fog}^{L^*} - \varepsilon)\log[re^{L(r)}]. \tag{13}$$

So combining (2) and (13) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[t]} M(r,fog)}{\log^{[t]} M(r^A,g)} \geq \frac{{}^{(t)}\rho_{fog}^{L^*} - \varepsilon}{A({}^{(t)}\rho_g^{L^*} + \varepsilon)}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r,fog)}{\log^{[t]} M(r^A,g)} \geq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A^{(t)}\rho_g^{L^*}}. \tag{14}$$

Thus the theorem follows from (12) and (14). □

Remark 2. Considering $f = z$, $g = \exp z$, $A = 1$, $t = 2$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ for any real number p, one can easily verify that the sign '$\leq$' cannnot be replaced by '$<$' only in Theorem 2.

Theorem 3. *Let f and g be two entire functions satisfying: (i) $0 <^{(t)} \rho_g^{L^*} < \infty$, (ii) $0 <^{(t)} \sigma_g^{L^*} < \infty$, (iii) ${}^{(t)}\rho_{fog}^{L^*} =^{(t)} \rho_g^{L^*}$, (iv) $0 <^{(t)} \sigma_{fog}^{L^*} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)} \leq \frac{{}^{(t)}\sigma_{fog}^{L^*}}{{}^{(t)}\sigma_g^{L^*}} \leq \limsup_{r\to\infty} \frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)},$$

where $t = 2, 3, ...$

Proof. From the definition of t-th generalised L^*-type of a composite entire function we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[t-1]} M(r,fog) \leq ({}^{(t)}\sigma_{fog}^{L^*} + \varepsilon)[re^{L(r)}]^{{}^{(t)}\rho_{fog}^{L^*}}. \tag{15}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[t-1]} M(r,g) \geq ({}^{(t)}\sigma_g^{L^*} - \varepsilon)[re^{L(r)}]^{{}^{(t)}\rho_g^{L^*}}. \tag{16}$$

As ${}^{(t)}\rho_{fog}^{L^*} =^{(t)} \rho_g^{L^*}$ from (15) and (16) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)} \leq \frac{{}^{(t)}\sigma_{fog}^{L^*} + \varepsilon}{{}^{(t)}\sigma_g^{L^*} - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)} \leq \frac{{}^{(t)}\sigma_{fog}^{L^*}}{{}^{(t)}\sigma_g^{L^*}}. \tag{17}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[t-1]} M(r,fog) \geq ({}^{(t)}\sigma_{fog}^{L^*} - \varepsilon)[re^{L(r)}]^{{}^{(t)}\rho_{fog}^{L^*}} \tag{18}$$

and for all sufficiently large values of r,

$$\log^{[t-1]} M(r,g) \le ({}^{(t)}\sigma_g^{L^*} + \varepsilon)[re^{L(r)}]^{{}^{(t)}\rho_g^{L^*}}. \tag{19}$$

By condition (iii) we obtain from (18) and (19) for a sequence of values of r tending to infinity that

$$\frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)} \ge \frac{{}^{(t)}\sigma_{fog}^{L^*} - \varepsilon}{{}^{(t)}\sigma_g^{L^*} + \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, we get from above

$$\limsup_{r\to\infty} \frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)} \ge \frac{{}^{(t)}\sigma_{fog}^{L^*}}{{}^{(t)}\sigma_g^{L^*}}. \tag{20}$$

Thus the theorem follows from (17) and (20). □

Remark 3. The sign '$\le$' in Theorem 3 cannnot be replaced by '$<$' only which is evident from the following example.

Example 2. Let $f = z$, $g = \exp z$, $t = 2$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any real number. So

$${}^{(2)}\rho_{fog}^{L^*} = {}^{(2)}\ \rho_g^{L^*} = 1 \quad \text{and} \quad {}^{(t)}\sigma_{fog}^{L^*} = {}^{(t)}\ \sigma_g^{L^*} = \frac{1}{\exp(\frac{1}{p})}.$$

Also

$$\limsup_{r\to\infty} \frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)} = 1 = \liminf_{r\to\infty} \frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)}.$$

Hence

$$\liminf_{r\to\infty} \frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)} = \frac{{}^{(t)}\sigma_{fog}^{L^*}}{{}^{(t)}\sigma_g^{L^*}} = 1 = \limsup_{r\to\infty} \frac{\log^{[t-1]} M(r,fog)}{\log^{[t-1]} M(r,g)}.$$

Theorem 4. *Let f, g, h be three entire functions such that $g \sim h$. Then*

$${}^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\ \rho_h^{L^*}(f) \quad \textit{and} \quad {}^{(t)}\lambda_g^{L^*}(f) = {}^{(t)}\ \lambda_h^{L^*}(f),$$

where $t = 1, 2, 3, ...$

Proof. Let $\varepsilon(>0)$ be choosen arbitrary. By Lemma 1 it follows for all large values of r that

$$G(r) < (1+\varepsilon)H(r) < H(\alpha r)\,, \tag{21}$$

where $\alpha > 1$ is such that $1 + \varepsilon < \alpha$.

From (21) we obtain for all large r that

$$\begin{aligned} r &< G^{-1}(H(\alpha r)) \\ \text{i.e.,}\quad \frac{1}{\alpha}H^{-1}(t) &< G^{-1}(t)\,, \quad \text{where } t = H(\alpha r) \end{aligned}$$

$$\text{i.e.,}\quad H^{-1}(r) \;<\; \alpha G^{-1}(r). \tag{22}$$

Now by (22) we get that

$$^{(t)}\rho_h^{L^*}(f) = \limsup_{r\to\infty} \frac{\log^{[t]} H^{-1}F(r)}{\log[re^{L(r)}]} \le \limsup_{r\to\infty} \frac{\log^{[t]}[\alpha G^{-1}F(r)]}{\log[re^{L(r)}]}$$

$$= \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(r)}{\log[re^{L(r)}]} =^{(t)} \rho_g^{L^*}(f). \tag{23}$$

Since $g \sim h$, therefore $h \sim g$ and in a similar manner one can verify that

$$^{(t)}\rho_g^{L^*}(f) \le^{(t)} \rho_h^{L^*}(f). \tag{24}$$

Thus from (23) and(24) it follows that

$$^{(t)}\rho_g^{L^*}(f) =^{(t)} \rho_h^{L^*}(f).$$

In a similar way one can show that

$$^{(t)}\lambda_g^{L^*}(f) =^{(t)} \lambda_h^{L^*}(f).$$

This proves the theorem. □

Remark 4. The converse of Theorem 4 is not true which is evident from the following example.

Example 3. Let $g = \exp z$, $h = \exp 2z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any real number. Also let $t = 1$.

Then $G(r) = \exp r$ and $H(r) = \exp 2r$, so that

$$\frac{G(r)}{H(r)} = \frac{\exp r}{\exp 2r} \to 0 \text{ as } r \to \infty.$$

Hence g *not* $\sim h$

But

$$^{(t)}\rho_g^{L^*}(f) =^{(t)} \rho_h^{L^*}(f) = 1 =^{(t)} \lambda_g^{L^*}(f) =^{(t)} \lambda_h^{L^*}(f).$$

Theorem 5. *Let f, g, h be three entire functions such that $g \sim h$. Then*

$$^{(t)}\rho_f^{L^*}(g) =^{(t)} \rho_f^{L^*}(h) \quad \textit{and} \quad ^{(t)}\lambda_f^{L^*}(g) =^{(t)} \lambda_f^{L^*}(h),$$

where $t = 1, 2, 3, \ldots$.

Proof. Since $g \sim h$, in view of Lemma 1 for $\varepsilon_1 > 0$, there exists $R_1 > 0$ such that

$$G(r) < (l + \varepsilon_1)H(r) < H(\beta r)\,, \tag{25}$$

where $0 < l < \infty$, $r > R_1$ and $\beta > 1$ such that $1 + \varepsilon_1 < \beta$.

Now from (25) we obtain that

$$\begin{aligned} {}^{(t)}\rho_f^{L^*}(g) &= \limsup_{r\to\infty} \frac{\log^{[t]} F^{-1}G(r)}{\log[re^{L(r)}]} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[t]} F^{-1}H(\beta r)}{\log[re^{L(r)}]}. \end{aligned} \quad (26)$$

For $0 < \varepsilon_2 < 1$ there exists $R_2 > 0$ such that for $r \geq R_2$

$$\log[re^{L(r)}] > (1-\varepsilon_2)\log[\beta re^{L(r)}]. \quad (27)$$

So from (26) and (27) we get that

$$\begin{aligned} {}^{(t)}\rho_f^{L^*}(g) &\leq \limsup_{r\to\infty} \frac{\log^{[t]} F^{-1}H(\beta r)}{(1-\varepsilon_2)\log[\beta re^{L(r)}]} \\ &= \frac{1}{1-\varepsilon_2}\ {}^{(t)}\rho_f^{L^*}(h). \end{aligned}$$

Since $0 < \varepsilon_2 < 1$ is arbitrary,

$${}^{(t)}\rho_f^{L^*}(g) \leq {}^{(t)}\rho_f^{L^*}(h). \quad (28)$$

As also $h \sim g$, we obtain in a like manner that

$${}^{(t)}\rho_f^{L^*}(h) \leq {}^{(t)}\rho_f^{L^*}(g). \quad (29)$$

Combining (28) and (29) it follows that

$${}^{(t)}\rho_f^{L^*}(g) = {}^{(t)}\rho_f^{L^*}(h).$$

Similarly we can show that

$${}^{(t)}\lambda_f^{L^*}(g) = {}^{(t)}\lambda_f^{L^*}(h).$$

Thus the theorem is proved. □

In the line of Theorem 4 and Theorem 5 we may state the following theorem without proof.

Theorem 6. *Let f, g, h, k be entire functions with $f \sim g$ and $h \sim k$. Then*

$${}^{(t)}\rho_h^{L^*}(f) = {}^{(t)}\rho_k^{L^*}(f) = {}^{(t)}\rho_h^{L^*}(g) = {}^{(t)}\rho_k^{L^*}(g)$$

and

$${}^{(t)}\lambda_h^{L^*}(f) = {}^{(t)}\lambda_k^{L^*}(f) = {}^{(t)}\lambda_h^{L^*}(g) = {}^{(t)}\lambda_k^{L^*}(g).$$

Theorem 7. *Let f, g and h be any three entire functions. If $F(r) \leq G(r)$ for all large r, then*

$${}^{(t)}\rho_h^{L^*}(f) \leq {}^{(t)}\rho_h^{L^*}(g),$$

where $t = 1, 2, 3, \ldots$.

Proof. In view of the condition $F(r) \leq G(r)$ we get that

$$\begin{aligned} {}^{(t)}\rho_h^{L^*}(f) &= \limsup_{r\to\infty} \frac{\log^{[t]} H^{-1}F(r)}{\log[re^{L(r)}]} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[t]} H^{-1}G(r)}{\log[re^{L(r)}]} \\ &= {}^{(t)}\rho_h^{L^*}(g). \end{aligned}$$

This proves the theorem. □

The following theorem can be carried out in the line of Theorem 7 and so the proof is omitted.

Theorem 8. *Let f, g and h be any three entire functions. If $G(r) \leq H(r)$ for all large r, then*

$${}^{(t)}\rho_g^{L^*}(f) \geq {}^{(t)}\rho_h^{L^*}(f),$$

where $t = 1, 2, 3, \dots$.

Theorem 9. *Let f and g be any two transcendental entire functions. Then*

$${}^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\rho_g^{L^*}(f'),$$

where $t = 1, 2, 3, \dots$ and f' denotes the derivative of f.

Proof. Let $\bar{F}(r) = \max_{|z|=r} |f'(z)|$. Without loss of any generality we may assume that $f(0) = 0$. Otherwise we set $f_1(z) = zf(z)$. Then $f_1(0) = 0$ and in view of Lemma 3 it follows that ${}^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\rho_g^{L^*}(f_1)$. We may write $f(z) = \int_0^z f'(t)\, dt$, where the line of integration is the segment from $z = 0$ to $z = re^{i\theta_0}$, $r > 0$. Let $z_1 = re^{i\theta_1}$ be such that $|f(z_1)| = \max_{|z|=r} |f(z)|$. Then

$$\begin{aligned} F(r) &= |f(z_1)| = \left| \int_0^{z_1} f'(t)\, dt \right| \\ &\leq r \max\{|f'(z)| : |z| = r\} \\ &= r\bar{F}(r). \end{aligned} \tag{30}$$

Let C denote the circle $|t - z_0| = r$, where z_0 is defined so that $|f'(z_0)| = \max_{|z|=r} |f'(z)|$. So

$$\bar{F}(r) = \max_{|z|=r} |f'(z)| = |f'(z_0)|$$

$$= \left| \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t-z_0)^2}\, dt \right|$$
$$\le \frac{1}{2\pi} \frac{F(2r)}{r^2} 2\pi r = \frac{F(2r)}{r}. \tag{31}$$

From (30) and (31) we obtain that
$$\frac{F(r)}{r} \le \bar{F}(r) \le \frac{F(2r)}{r} \text{ for } r > 0. \tag{32}$$

Let $\sigma \in (0,1)$. Since f is transcendental, from the second part of Lemma 2 it follows that
$$\lim_{r\to\infty} \frac{F(r^s)}{r^n F(r)} = \infty \ (s > 1).$$

We set $s = \frac{1}{\sigma}$ and so $F(r^s) > r^n F(r)$ for all large r. If we replace r by r^σ, then from above we get that
$$F(r^{s\sigma}) > r^{n\sigma} F(r^\sigma) \ge rF(r^\sigma),$$
where n is such that $n\sigma \ge 1$, i.e., $F(r) > rF(r^\sigma)$.

From (32) it follows that
$$F(r^\sigma) < \frac{F(r)}{r} \le \bar{F}(r) \le \frac{F(2r)}{r} < F(2r),\ r > 1,$$
$$\text{i.e., } F(r^\sigma) < \bar{F}(r) < F(2r) \tag{33}$$
and so
$$\limsup_{r\to\infty} \frac{\log^{[t]} G^{-1} F(r^\sigma)}{\log[re^{L(r)}]} \le \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1} \bar{F}(r)}{\log[re^{L(r)}]} \le \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1} F(2r)}{\log[re^{L(r)}]},$$
$$\text{i.e., } \sigma^{(t)}\rho_g^{L^*}(f) \le^{(t)} \rho_g^{L^*}(f') =^{(t)} \rho_g^{L^*}(f'). \tag{34}$$

Letting $\sigma \to 1-0$, we obtain that
$${}^{(t)}\rho_g^{L^*}(f) =^{(t)} \rho_g^{L^*}(f').$$

Thus the theorem is proved. □

In the line of Theorem 9 we may state the following theorem without poof.

Theorem 10. *If f and g are any two transcendental entire functions, then*
$${}^{(t)}\rho_{g'}^{L^*}(f) =^{(t)} \rho_{g'}^{L^*}(f'),$$
where $t = 1, 2, 3, \ldots$ and f', g' respectively denote the derivatives of f and g.

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