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OSCILLATIONS FOR SECOND ORDER NEUTRAL NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract: Some new oscillation criteria are obtained for the second order neutral nonlinear differential equation of the form

$$[r(t)[y(t)+p(t)y(\sigma(t))]']' + \sum_{i=1}^{n} q_i(t) f_i(y(\tau_i(t))) = 0, \quad t \geq t_0 \,.$$

Our results improve some known results and show that the oscillation of some second order linear ordinary differential equations implies the oscillation of the nonlinear second order neutral delay differential equation.

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1. Introduction

In this paper we consider the oscillation behavior of solutions of the second order neutral nonlinear differential equation

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$$\left[r(t)[y(t)+p(t)y(\sigma(t))]'\right]'+\sum_{i=1}^{n}q_i(t)f_i\left(y(\tau_i(t))\right)=0\,, \qquad (1)$$

where $t \geq t_0$ and the functions $p, \sigma, \tau_i, q_i, f_i$ $(i = 1, 2, ..., n)$ are to be specified in the following text.

By a solution of equation (1), we mean a continuously differentiable function $y(t)$ which is defined for $t \geq \min\{\sigma(t_0), \tau_i(t_0), i = 1, 2, ..., n\}$ such that $y(t))$ satisfies (1) for all $t \geq t_0$. In the sequel, it will be always assumed that solutions of equation (1) exist on some half-line $[T, \infty)(T \geq t_0)$. We restrict our attention only to the nontrivial solution $y(t)$ of (1), i.e., to the solution $y(t)$ such that $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq t_0$. A nontrivial solution of equation (1) is called oscillatory if it has arbitrary large zeros, otherwise, it is called nonoscillatory. equation (1) is called oscillatory if all its solutions are oscillatory.

The oscillation problem for linear ordinary equations

$$y''(t)+q(t)y(t)=0, \quad t \geq t_0\,, \qquad (2)$$

$$\left[p(t)y'(t)\right]'+q(t)y(t)=0, \quad t \geq t_0\,, \qquad (3)$$

as well as for the nonlinear ordinary differential equation

$$\left[p(t)y'(t)\right]'+q(t)f\left(y(t)\right)=0, \quad t \geq t_0, \qquad (4)$$

and the nonlinear delay equations

$$\left[p(t)y'(t)\right]'+q(t)f\left(y(\tau(t))\right)=0, \quad t \geq t_0, \qquad (5)$$

$$[r(t)[y(t)+p(t)y(\sigma(t))]']'+q(t)f(y(\tau(t)))=0,\, t \geq t_0, \qquad (6)$$

has been studied by many authors with different methods. Some result can be fond in [15], [1], [13], [12], [10], [3], [2], [7], [11], [8] and references therein.

In one hand, Wintner [15] used a so-called averaging technique and proved that if $q \in C([t_0, \infty), R)$ and satisfies

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(v)dvds=\infty\,, \qquad (7)$$

then equation (2) is oscillatory.

This technique was explored further by Kamenev [7] who prove that if

$$\limsup_{t\to\infty} t^{1-m}\int_{t_0}^{t}(t-s)^{m-1}q(s)ds=\infty \text{ for } m>2\,, \qquad (8)$$

then equation (2) is oscillatory.

Philos [11] further improved the results of the Kamenev by proving the following:

Theorem A. *Suppose there exist continuous functions* $H, h : D = \{(t, s) :$

$t \geq s \geq t_0\} \to R$ *such that:*

(H_1) $H(t,t) = 0, t \geq t_0, H(t,s) > 0, t > s \geq t_0$.

(H_2) *H has a continuous and nonpositive partial derivative on D with respect to the second variable and satisfies*

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \geq 0. \tag{9}$$

Further, suppose that

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right] ds = \infty. \tag{10}$$

Then equation (2) is oscillatory.

For the second order neutral delay differential equations, Grammatikopoulos et al [4] obtained that if $0 \leq p(t) \leq 1, q(t) \geq 0$ and $\int_{t_0}^{\infty} q(s)[1-p(s-\sigma)]ds = \infty$, then

$$(y(t) + p(t)y(t-\tau))'' + q(t)y(t-\sigma) = 0\,. \tag{11}$$

is oscillatory.

In [14], by using Riccati technique and averaging functions method, Ruan extends Philos oscillation results for second order neutral delay differential equation

$$[a(t)(y(t) + p(t)y(t-\tau))']' + q(t)f((t-\sigma)) = 0, \tag{12}$$

In [9], by using a generalized Riccati transformation, Li improved the results of Run for equation (12). In [17], Zhuang and Li studied the oscillation of the equation

$$[y(t) + p(t)y(\sigma(t))]'' + \sum_{i=1}^{n} q_i(t)f_i\left(y(\tau_i(t))\right) = 0 \tag{13}$$

by using the generalized Riccati technique and averaging technique.

In the other hand, Huang [4] presented the following interval criteria for oscillation of equation (2).

Theorem B. *If there exists $t_0 \geq 0$ such that for every positive integer n,*

$$\int_{2^n t_0}^{2^{(n+1)}t_0} q(t)dt \geq \frac{\alpha}{2^n t_0}, \tag{14}$$

where $\alpha > \alpha_0 = 3 - 2\sqrt{2}$, then equation (2) is oscillatory.

By replacing the sequence $\{2^n\}$ in Theorem B by $\{\lambda^n\}$ with $\lambda > 1$, Wong [16] generalized Huangs criteria as follows:

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Theorem C. *Let $\lambda > 1$ and $\alpha_0 = (\sqrt{\lambda} - 1)^2$. If there exist $t_0 > 0$ and $\alpha > \alpha_0$ such that for all $n \in N_0$,*

$$\int_{\lambda^n t_0}^{\lambda^{(n+1)} t_0} q(t)dt \geq \frac{\alpha}{(\lambda - 1)\lambda^n t_0}, \tag{15}$$

then equation (2) is oscillatory.

The purpose of this paper is to establish some new oscillation criteria by generalizing Theorem A and applying Theorem B and Theorem C to the more general neutral delay differential equation (1). Our results improve and generalized Theorem A, some Huang's type oscillation criteria are also obtained. At the same time, our results show that the oscillation of some second order linear ordinary differential equations implies the oscillation of the nonlinear second order neutral delay differential equation (1), thus we can obtain some new oscillation theorems for equation (1), which do not need the condition of the integrals of the coefficient.

In what follows, we always assume that:

(C_1) the function $r : [t_0, \infty) \to (0, \infty), r'(t) \geq 0, \int^{\infty} \frac{1}{r(t)} dt = \infty$;

(C_2) the function $p : [t_0, \infty) \to [0, 1)$ is a continuous function;

(C_3) the functions $q_i : [t_0, \infty) \to [0, \infty), i = 1, 2, ..., n,$ are continuous and $q_i \not\equiv 0$ on any $[T, \infty)$ for some $T \geq t_0$;

(C_4) the functions $f_i : R \to R$ are continuous and $\frac{f_i(y)}{y} \geq \mu_i > 0$ or $f_i'(y) \geq \mu_i > 0$ for $y \neq 0, i = 1, 2, ..., n,$ μ_i are constants;

(C_5) the function $\sigma : [t_0, \infty) \to R$ is continuous and nondecreasing, $\sigma(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \sigma(t) = \infty$;

(C_6) the functions $\tau_i : [t_0, \infty) \to R$ are continuous, $\tau_i \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau_i(t) = \infty, i = 1, 2, ..., n$.

2. Main Results

First we give two lemmas which will be used in the following results.

Lemma 2.1. *If $y(t)$ is a nonoscillation solution of equation (1), then $z(t)z'(t)$ is eventually positive, where $z(t) = y(t) + p(t)y(\sigma(t))$ for $t \geq t_0$.*

Proof. Without loss of generality we may assume that $y(\sigma(t)) > 0$ for $t \geq T_1 \geq t_0$, where T_1 is a positive number. Then $z(t) > 0$ and

$$[r(t)z'(t)]' = -\sum_{i=1}^{n} q_i(t)f_i(y(\tau_i(t))) \leq 0,\ t \geq T_1. \tag{16}$$

Therefore, $r(t)z'(t)$ is decreasing. We claim that $z'(t) > 0$ for $t \geq T_1$. If it is not the case, we suppose that there exists a real number $T_2 \geq T_1$ such that $z'(T_2) \leq 0$. We can see that

$$r(t)z'(t) \leq r(T_2)z'(T_2), \quad t \geq T_2,$$

that is

$$z'(t) \leq \frac{1}{r(t)}r(T_2)z'(T_2), \quad t \geq T_2\,.$$

Integrating above inequality from T_2 to t, we have

$$z(t) \leq z(T_2) + r(T_2)z'(T_2)\int_{T_2}^{t} \frac{1}{r(t)}dt,$$

considering about (C_1), the above inequality implies that $z(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $z(t) > 0$ for $t \geq T_1$. This completes our proof. □

Lemma 2.2. *Assume that* $y(t) \in C^2[t_0.\infty)$ *satisfies*

$$y(t) > 0,\ y'(t) > 0,\ y''(t) \leq 0,\ t \geq t_0,$$

then for each $0 < l_i < 1$, *there exists a* $T_2 \geq t_0$ *such that*

$$y(\tau_i(t)) \geq l_i y(t)\frac{\tau_i(t)}{t},\ t \geq T_2,\ i = 1, 2, ..., n.$$

The proof is similar to that of [1], Lemma 2.1, here we omit it.

Theorem 2.1. *Suppose that there exist a function* $\phi \in C^1([t_0, \infty), R^+)$ *and a function* $F \in C([t_0, \infty), R)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\psi(s) - \frac{1}{4}r(s)\Phi(s)F^2(s)\right] e^{\int_{t_0}^{s} F(\tau)d\tau} ds = \infty\,, \tag{17}$$

where $\Phi(s) = e^{-2\int^{s} \phi(\tau)d\tau}$, *and*

$$\psi(s) = \Phi(t)\left\{\sum_{i=1}^{n} \mu_i l_i q_i(t)\frac{\tau_i(t)}{t}[1 - p(\tau_i(t))] + r(t)\phi^2(t) - [r(t)\phi(t)]'\right\}.$$

Then equation (1) is oscilllatory.

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1). By Lemma 2.1, $z(t)z'(t)$ is eventually positive. Without loss of generality, we assume that $y(t) > 0, z(\tau_i(t)) > 0.z'(t) > 0$ and $z(\sigma(\tau_i(t))) > 0, i = 1, 2, ..., n$, for $t \geq T_1 \geq$

t_0. Since the case when $y(t)$ is eventually negative can be treated analogously. Hence by Lemma 2.2, for any $0 < l_i < 1$ there exists a $T_2 \geq T_1$ such that

$$z(\tau_i(t)) \geq l_i z(t)\frac{\tau_i(t)}{t}, \quad i = 1, 2, ..., n, \quad t \geq T_2. \tag{18}$$

Now, observing equation (1), we have

$$[r(t)z'(t)]' + \sum_{i=1}^{n} q_i(t) f_i(y(\tau_i(t))) = 0. \tag{19}$$

Using $(C_1), (C_2), (C_3)$ and (19), we get

$$[r(t)z'(t)]' + \sum_{i=1}^{n} \mu_i q_i(t) \left[z(\tau_i(t) - p(\tau_i(t))z(\sigma(\tau_i(t)))\right] \leq 0. \tag{20}$$

In view of the fact that $z(t) \geq y(t)$ and $z'(t) > 0$, yield

$$r'(t)z'(t) + r(t)z''(t) + \sum_{i=1}^{n} \mu_i q_i(t) \left[1 - p(\tau_i(t))\right] z(\tau_i(t)) \leq 0. \tag{21}$$

Difine

$$w(t) = \Phi(t) \left\{ \frac{r(t)z'(t)}{z(t)} + r(t)\phi(t) \right\}, \quad t \geq T = \max\{T_0, T_1, T_2\}. \tag{22}$$

Then

$$\begin{aligned} w'(t) &= -2\phi(t)w(t) + \Phi(t) \left\{ \frac{[r(t)z'(t)]'}{z(t)} - \frac{1}{r(t)} \left[\frac{r(t)z'(t)}{z(t)}\right]^2 + [r(t)\phi(t)]' \right\} \\ &\leq -2\phi(t)w(t) + \Phi(t) \Bigg\{ -\sum_{i=1}^{n} \mu_i l_i q_i(t) \frac{\tau_i(t)}{t} [1 - p(\tau_i(t))] \\ &\quad -\frac{1}{r(t)} \left[\frac{w(t)}{\Phi(t)} - r(t)\phi(t)\right]^2 + [r(t)\phi(t)]' \Bigg\} \\ &= -\psi(t) - \frac{w^2(t)}{r(t)\Phi(t)}, \end{aligned} \tag{23}$$

i.e.,

$$w'(t) \leq \psi(t) - \frac{w^2(t)}{r(t)\Phi(t)},$$

where

$$\psi(t) = \Phi(t) \left\{ \sum_{i=1}^{n} \mu_i l_i q_i(t) \frac{\tau_i(t)}{t} [1 - p(\tau_i(t))] + r(t)\phi^(t) - [r(t)\phi(t)]' \right\}.$$

Hence, for all $t \geq T$, we have

$$w'(t) \leq -\left[\psi(t) - \frac{1}{4}r(t)\Phi(t)F^2(t)\right] - \left[\frac{w^2(t)}{r(t)\Phi(t)} + \frac{1}{4}r(t)\Phi(t)F^2(t)\right],$$

i.e.,

$$w'(t) + F(t)w(t) \leq -\left[\psi(t) - \frac{1}{4}r(t)\Phi(t)F^2(t)\right],$$

$$w(t)e^{\int_{t_0}^{t} F(\tau)d\tau} - w(T)e^{\int_{t_0}^{T} F(\tau)d\tau} \leq -\int_T^t \left[\psi(s) - \frac{1}{4}r(s)\Phi(s)F^2(s)\right] e^{\int_{t_0}^{s} F(\tau)d\tau} ds,$$

Hence

$$\int_T^t \left[\psi(s) - \frac{1}{4}r(s)\Phi(s)F^2(s)\right] e^{\int_{t_0}^{s} F(\tau)d\tau} ds \leq w(T)e^{\int_{t_0}^{T} F(\tau)d\tau} - w(t)e^{\int_{t_0}^{t} F(\tau)d\tau}. \quad (24)$$

In view of $w(t) \geq 0$, we get

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\psi(s) - \frac{1}{4}r(s)\Phi(s)F^2(s)\right] e^{\int_{t_0}^{s} F(\tau)d\tau} ds \leq w(T)e^{\int_{t_0}^{T} F(\tau)d\tau}, \quad (25)$$

which contracdicts the assumption (17).

This completes the proof of Theorem 2.1. □

Taking $F(s) = \frac{\partial H(t,s)}{\partial s}/H(t,s)$ and $\phi \geq 0$ where $H(t,s)$ is defined as in Theorem A, and from (24), we have the following Philo's type theorem.

Theorem 2.2. *Suppose that there exists a function $\phi \in C^1([t_0,+\infty), R^+)$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\psi(s) - \frac{1}{4}\Phi(s)r(s)h^2(t,s)\right] ds = \infty. \quad (26)$$

Then equation (1) is oscillatory.

Next we present another oscillation theorem.

Theorem 2.3. *If the following ordinary differential equation*

$$y'' + Q(t)y(t) = 0, \quad (27)$$

where

$$Q(t) = \frac{1}{r(t)}\left\{\sum_{i=1}^{n} \mu_i l_i q_i(t)\frac{\tau_i(t)}{t}[1 - p(\tau_i(t))] + \frac{r'^2(t)}{4r(t)} - \frac{r''(t)}{2}\right\}, \quad (28)$$

is oscillatory, then equation (1) is oscilllatory.

Proof. Let $y(t)$ be a non-oscillatory solution of equations (1). Without loss of generality, We assume that $y(t) > 0, t \geq T_0 \geq t_0$. Similar to the proof of Theorem 2.1, we can get

$$w'(t) \leq -Q(t) - w^2(t) \quad \text{for} \quad t \geq T_0 \quad (29)$$

where $Q(t)$ is defined as in (28). In fact, we taking $\phi(t) = \frac{r'(t)}{2r(t)}$ in Theorem 2.1, we obtain (29) from (23).

Therefore, from (29), by using Theorem 7.2 in [5], Chapter XI, we see that equation (27) is non-oscillatory. This contradicts the fact that equation (27) is oscillatory.

The proof of Theorem 2.3 is completed. □

Corollary 2.1. *Assume that*

$$\infty \geq \lim_{r\to\infty} t^2 \frac{1}{r(t)} \left\{ \sum_{i=1}^{n} \mu_i l_i q_i(t) \frac{\tau_i(t)}{t} [1 - p(\tau_i(t))] + \frac{r'^2(t)}{4r(t)} - \frac{r''(t)}{2} \right\} > \frac{1}{4}, \quad (30)$$

then equations (1) is oscillatory.

Proof. From Theorem 2 and Theorem 7.1 in [5], Chapter XI, it is easy to see that the result of Corollary 2.1 is true. □

Corollary 2.2. *Assume that*

$$\infty \geq \liminf_{r\to\infty} t \int_t^{\infty} \frac{1}{r(t)} \left\{ \sum_{i=1}^{n} \mu_i l_i q_i(t) \frac{\tau_i(t)}{t} [1 - p(\tau_i(t))] + \frac{r'^2(t)}{4r(t)} - \frac{r''(t)}{2} \right\} dt > \frac{1}{4}, \quad (31)$$

then equations (1) is oscillatory.

Corollary 2.3. *If there exist $T > t_0$ and $\alpha > 3 - 2\sqrt{2}$ such that for every $n \in N$,*

$$\int_{2^n T}^{2^{n+1} T} \frac{1}{r(t)} \left\{ \sum_{i=1}^{n} \mu_i l_i q_i(t) \frac{\tau_i(t)}{t} [1 - p(\tau_i(t))] + \frac{r'^2(t)}{4r(t)} - \frac{r''(t)}{2} \right\} dt > \frac{\alpha}{2^n T}, \quad (32)$$

then equation (1) is oscillatory.

Corollaries 2.2 and 2.3 are easy to be proved by Theorem 2.2 of this paper and Theorems A and 2 of Huang [16].

Corollary 2.4. *Let $\lambda > 1$ and $\alpha_0 = (\sqrt{\lambda} - 1)^2$. If there exist $T > t_0$ and $\alpha > \alpha_0$ such that for every $n \in N$,*

$$\int_{\lambda^n T}^{\lambda^{n+1} T} \frac{1}{r(t)} \left\{ \sum_{i=1}^{n} \mu_i l_i q_i(t) \frac{\tau_i(t)}{t} [1 - p(\tau_i(t))] + \frac{r'^2(t)}{4r(t)} - \frac{r''(t)}{2} \right\} dt > \frac{\alpha}{(\lambda - 1)\lambda^n T}, \quad (33)$$

then equation (1) is oscillatory.

Remark 1. From Theorem 2.3, we can get more results for the oscillation of equation (1) similar to Corollaries 2.2, 2.3 and 2.4.

Remark 2. Corollaries 2.3 and 2.4 are different from most known ones in the sense that they are based on the information only on a sequence of intervals like $[2^n T, 2^{n+1}T]$, rather than on the whole half-line $[t_0, \infty)$.

Example 1. Consider the following equation

$$[x(t) + (1 - e^{-\mu t})x(t-\tau)]'' + \sum_{i=1}^{n} \frac{a_i}{t^2} e^{\lambda_i \mu t} x(\lambda_i t) = 0, \tag{34}$$

where $\mu, \lambda_i, a_i > 0, \tau \geq 0, 0 < \lambda_i < 1, i = 1, 2, \cdots, n, t \geq 1$. Here $p(t) = 1 - e^{\mu t}, q_i(t) = \frac{a_i}{t^2} e^{\lambda_i \mu t}, \tau_i(t) = \lambda_i t$. Then we have

$$Q(t) = \sum_{i=1}^{n} l_i q_i(s) \frac{\tau_i(t)}{t} (1 - p(\tau_i(s))) = \sum_{i=1}^{n} \frac{a_i}{t^2} e^{\lambda_i \mu s} \lambda_i l_i e^{-\mu \lambda_i s} = \sum_{i=1}^{n} \lambda_i l_i a_i \frac{1}{t^2}.$$

Then, for each $0 < \lambda_i < 1, t \geq 1$,

$$\lim_{t \to \infty} t^2 Q(t) = \sum_{i=1}^{n} \lambda_i l_i a_i.$$

Hence, by Corollary 2.1, equation (34) is oscillatory if $\sum_{i=1}^{n} \lambda_i l_i a_i > \frac{1}{4}$. However, the main results of [1], [13], [12], [10], [3], [2], [11], [4], [14], fail to apply to equation (34), since $(1 - e^{-\mu t})$ is not equal to zero, or $\int^{\infty} q_1(t)(1 - p(\lambda_1 t))dt = \int^{\infty} \frac{\alpha_1}{t^2} dt < \infty$ $(n = 1)$.

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