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ON THE FAILURE OF CLIFFORD'S INEQUALITY FOR LINE BUNDLES ON REDUCIBLE CURVES

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Abstract: We give a cheap way to construct pairs (X, L) with X a stable curve and $L \in \mathrm{Pic}(X)$ such that L is ample and spanned, $\deg(L) \le 2p_a(X) - 2$ and $h^0(X, L) > \deg(L)/2 + 1$, i.e. L does not satisfy Clifford's inequality.

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Let X be any nodal projective curve. Let $\mathcal{B}(X)$ denote the set of the irreducible components of X. The *dual graph* $\|X\|$ of X is the following non-oriented graph with multiple edges, but no loop. There is a bijection between the set of all vertices of $\|X\|$ and $\mathcal{B}(X)$. If $v \ne w$ are vertices of $\|X\|$ and T_v, T_w are the associated irreducible components of X, then v and w are connected by $\sharp(T_v \cap T_w)$ edges. Notice that the graph $\|X\|$ ignores the singularities of X lying on a unique irreducible component of X. The curve X is connected if and only if the graph $\|X\|$ is connected. Let X be a connected projective curve. We say that Clifford's inequality fails for $L \in \mathrm{Pic}(X)$ if $\deg(L) \le -2\chi(\mathcal{O}_X)$ and $h^0(X, L) > \deg(L)/2 + 1$. Here we prove the following result.

Theorem 1. *Let Y be a connected nodal curve such that $r := \sharp(\mathcal{B}(Y)) \ge 2$. Fix an ordering of the irreducible components $Y_1, \dots, Y_r$. Then there is a connected nodal curve X with the following properties:*

(a) $r = \sharp(\mathcal{B}(X))$ and there is an ordering $X_1, \dots, X_r$ of $\mathcal{B}(X)$ such that $X_i \cong Y_i$ for all $i \ne x$;

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(b) for every $i \in \{1, \dots, r\}$ identify the vertex of $\|X\|$ corresponding to X_i with the vertex corresponding to Y_i; with this identification we get $\|X\| = \|Y\|$;

(c) there is an ample and spanned $L \in Pic(X)$ such that $\deg(L) \le -2\chi(\mathcal{O}_X)$ *and* $h^0(X, L) > \deg(L)/2 + 1$.

Moreover, if Y is semistable (resp. stable), then we may find a semistable (resp. stable) curve X with properties (a), (b) and (c).

For the far more difficult construction of pairs (X, L) for which Clifford's inequality fails and L is balanced, see [1] and [2].

Proof of Theorem 1. Set $A := \cup_{i=1}^{r-1} Y_i$, $S := Y_r \cap A$ and $s := \sharp(S)$. Since Y is connected, $s > 0$. Since Y is nodal, S is an effective Cartier divisor of A. Fix an integer $z \ge \max\{3, s\}$ and any very ample line bundle R on A such that $h^0(A, R) = z$, $h^0(A, R(-S)) = z - s$ and $h^1(A, R) = 0$. Riemann-Roch gives $\deg(M) = z + \chi(\mathcal{O}_A)$ even if A is not connected. If $s \le 2$ set $w := 2$ and take as q_w any positive integer ≥ 2. If $s \ge 3$, then fix an integer $w \ge 2s$ and set $q_w := (w - 3n)^2/2n$. If $s \le 2$ set $w := 2$, take as X_r any smooth hyperelliptic curve of genus ≥ 2 and set $q_2' = q_2 = p_a(X_r)$. In this case let $M \in \mathrm{Pic}^2(X_r)$ be the hyperelliptic line bundle on X_r. Hence $\deg(R) = h^0(X_r, M) = 2$. The theory of curves in $\mathbb{P}^{s-1}$ with maximal genus shows the existence of a smooth and connected curve X_r of genus $q_w' \ge q_w$ and a morphism $\phi : X \to \mathbb{P}^{s-1}$ birational onto its image such that $\deg(\phi(X_r)) = w$ and $\phi(X_r))$ spans $\mathbb{P}^{s-1}$ ([3], Theorem III.3.7). Set $M := \phi^*(\mathcal{O}_{\mathbb{P}^{s-1}}(1)) \in \mathrm{Pic}^w(X_r)$. By construction M is spanned. Since $\phi(X_r))$ spans $\mathbb{P}^{s-1}$, $h^0(X_r, M) \ge s$. Fix a general $S' \subset X_r$ such that $\sharp(S') = s$. Since $h^0(X_r, M) \ge \sharp(S')$ andX_r is irreducible, $h^0(X_r, M(-S')) = h^0(X_r, M) - s$. Fix an ordering $P_1, \dots, P_s$ of S and an ordering $Q_1, \dots, Q_s$ of S'. Let X be the nodal curve obtained from $A \sqcup X_r$ gluing together each point $P_i \in A$ with the point $Q_i \in X_r$. We have $\|X\| = \|Y\|$ and $g := p_a(X) = -\chi(\mathcal{O}_A) + x - 2 + q_w'$. The integers $\chi(\mathcal{O}_A)$ and s are fixed.

Since X is a reduced curve, there is at least one line bundle L on X such that $L|A \cong R$ and $L|X_r \cong M$. Fix any such line bundle L. Since M and R are ample, L is ample. Consider the Mayer-Vietoris exact sequence:

$$0 \to L \to L|A \oplus L|X_r \to L|X_r \cap A \to 0\,. \tag{1}$$

Since $h^0(A, R(-S)) = z - s$ and $L|A \cong R$, the restriction map $H^0(A, L|A) \to H^0(X_r \cap A, L|X_r \cap A)$ is surjective. Hence (1) gives $h^0(X, L) = h^0(X_r, M) + z - s \ge z$. We have $\deg(L) = \deg(L|A) + \deg(L|X_r) = z + \chi(\mathcal{O}_A) + w$. If $s \le 1$ (and hence $w = 2$), then we may obviously find q_2 so large (depending only from z) that $2 + z \le 2g - 2$. Since $\deg(L) = 2 + z$ and $h^0(X, L) \ge z - s$, the inequality $h^0(X, L) > \deg(L/2) + 1$ is satisfied if $z - s > z/2 + 1$. Hence in this case we may take any $z \geq 2s + 5$. Now assume $s \geq 3$. To get simultaneously $2 + z \leq 2g - 2$ and $z > (z + \chi(\mathcal{O}_A) + w)/2 + 1$ it is sufficient to take z, w, q_w satisfying the bollowing inequalities:

$$z \geq \chi(\mathcal{O}_A) + w + 3\,, \tag{2}$$

$$w \geq 2s,\ 2 + z \leq (w - 3s + 3)/2(s - 1) - 2\,. \tag{3}$$

Since $\chi(\mathcal{O}_A)$ is a fixed integer, for $w \gg 0$ we may find z, w satisfying (2) and (3) and hence giving a pair (X, L) for which Clifford's inequality fails. Since the restriction map $H^0(A, L|A) \to H^0(X_r \cap A, L|X_r \cap A)$ is surjective, (1) gives the surjectivity of the restriction map $H^0(X, L) \to H^0(X_r, M)$. Since M is spanned, L is spanned at each point of X_r. Since the restriction map $H^0(X_r, L|X_r) \to H^0(X_r \cap A, L|X_r \cap A)$ is surjective, (1) gives the surjectivity of the restriction map $H^0(X, L) \to H^0(A, R)$. Since RM is spanned, L is spanned at each point of A. Hence L is spanned.

By construction if Y is semistable or stable, then X has the same property. □

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