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CLIFFORD'S THEOREM FOR RANK 2 SHEAVES ON CERTAIN STABLE CURVES

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Abstract: Here we prove Clifford's inequality for rank 2 torsion free sheaves on certain stable curves (e.g. the the binary curves).

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A key elementary result in the Brill-Noether theory of smooth projective curves is Clifford's Theorem and its several generalization. Clifford's Theorem is true for semistable vector bundles on a smooth curve ([1], Theorem 2.1, [7]). The situation is more complicated on reducible curves, even on stable or quasi-stable curves, even in the rank 1 case: in a few cases it holds (e.g. always on binary curves ([3]), for very low degree in many cases ([4], Section 4), but it fails in many other cases ([4], Section 4.3). Let X be a stable curve of genus $g \geq 2$. We recall that the notion of balanced line bundles on all quasi-stable curves with X as stable reduction is equivalent to the notion of ω_X-semistable sheaf on X with depth 1 and pure rank 1 ([6], Theorem 10.4.1). Here we show how to reduce a Clifford's Theorem for a spanned rank 2 ω_X-semistable sheaf E on X to a Clifford's Theorem for the rank 1 sheaf $\det(E)$. A pure rank r sheaf on X satisfies Clifford's inequality if $h^0(X, E) \leq r + \deg(E)/2$. By Riemann-Roch we need to assume that $\deg(E) + r\chi(\mathcal{O}_X)$ is very low. Serre duality shows that

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it is natural to assume $\deg(E) \le r(2g-2)$, where $g := p_a(X)$.

Here we prove the following result.

Theorem 1. *Let X be a reduced and connected projective curve with arithmetic genus g. Fix a polarization H on X. Assume that Clifford's inequality is true for every H-semistable depth 1 sheaf on X with pure rank 1 and degree $\le 2g-2$. Let E be a spanned depth 1 sheaf on X with pure rank 2 such that $\deg(E) \le 4g-4$.*

(a) If $\deg(E) \le 2g-2$ and $\det(E)$ is H-semistable, then E satisfies Clifford's inequality.

(b) If $\deg(E) \le 4g-4$, $H = \omega_X$ and $\det(E)$ is H-semistable, then E satisfies Clifford's inequality.

Proof. Since E is H-semistable, $h^0(X, E) = 0$ if $\deg(E) < 0$. Hence we may assume $\deg(E) \ge 0$. Since E is spanned and it has pure rank 2, it fits in an exact sequence

$$0 \to \mathcal{O}_X \to E \to \det(E) \to 0\,. \tag{1}$$

Hence $h^0(X, E) \le 1 + h^0(X, \det(E))$. If $\deg(E) \le 2g-2$, we may apply to $\det(E)$ the assumption on the rank 1 sheaves on X. Hence (a) is true. Now assume $H = \omega_X$ and $\deg(E) \le 4g-4$. By Riemann-Roch E satisfies Clifford's inequality if and only if $\omega_X \otimes E^*$ satisfies Clifford's inequality. Since X is Gorenstein, it is easy to check that E is H-semistable if and only if E^* is H-semistable. Similarly, $\det(E)$ is H-semistable if and only if $\det(E)^*$ is H-semistable. If $H = \omega_X$ (or it induces a proportional polarization), then $E^* \otimes \omega_X$ is H-semistable if and only if E^* is H-semistable and the same is true for the associated determinant. Hence to prove part (b) we may assume $2g-2 < \deg(E) \le 4g-4$. Hence $\deg(E \otimes \omega_X) < 2g-2$. We just saw that we may apply part (a) to $\omega_X \otimes E^*$ and get part (b). □

Remark 1. The proof just given shows that we only need to assume that either $\det(E)$ (case $\deg(E) \le 2g-2$) or $\omega_X^{\otimes 2} \otimes \det(E)^*$ (case $2g-2 \le \deg(E) \le 4g-4$) satisfies Clifford's inequality.

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