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AN ANALYSIS OF A COVOLUME METHOD FOR THE STATIONARY NAVIER-STOKES EQUATIONS

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Abstract: We introduce a covolume method for approximating the stationary Navier-Stokes equations and analyze the convergence of the covolume approximation. The covolume method uses the primal and dual partitions. The velocity is approximated using nonconforming piecewise linear functions and the pressure piecewise constants. We use an abstract theory to the study of the convergence of the covolume method for the Navier-Stokes equations, which is based on the results of approximation for branches of nonsingular solutions of nonlinear problems presented in [10]. Numerical results using a simple Picard type of iteration for solving the discrete Navier-Stokes equations are provided.

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Key Words: covolume method for approximating, stationary Navier-Stokes equations, numerical results

1. Introduction

Let Ω be a bounded polygonal domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary Γ. The stationary Navier-Stokes equations with the Dirichlet boundary condition are:

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$$-\nu\,\Delta \mathbf{u} + \mathbf{u}\cdot\nabla\,\mathbf{u} + \mathbf{grad}\,p = \mathbf{f} \qquad \text{in} \;\; \Omega, \tag{1.1}$$
$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in} \;\; \Omega, \tag{1.2}$$
$$\mathbf{u} = 0 \qquad \text{on} \;\; \Gamma, \tag{1.3}$$

where $\mathbf{u} : \Omega \to \mathbb{R}^2$ is the velocity field and $p : \Omega \to \mathbb{R}$ is the pressure; $\mathbf{f}$ represents the body force and $\nu > 0$ is the viscosity.

If the velocity of the flow is assumed sufficiently small, the convection term $\mathbf{u}\cdot\nabla\mathbf{u}$ is ignored. Then we have the Stokes problem:

$$-\nu\,\Delta \mathbf{u} + \mathbf{grad}\,p = \mathbf{f} \qquad \text{in} \;\; \Omega, \tag{1.4}$$
$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in} \;\; \Omega, \tag{1.5}$$
$$\mathbf{u} = 0 \qquad \text{on} \;\; \Gamma. \tag{1.6}$$

Let $H^i(\Omega), i = 1, 2$ be the usual Sobolev spaces, $H_0^1(\Omega)$ be the space of weakly differentiable functions with zero trace, and $L_0^2(\Omega)$ be the set of all L^2 functions over Ω with zero integral mean. $\mathbf{H}^i(\Omega), i = 1, 2$ consists of vector valued functions each of whose components belongs to $H^i(\Omega)$.

We now define the bilinear forms

$$a_0(\mathbf{u},\mathbf{v}) = \nu \int_\Omega \mathbf{grad}\;\mathbf{u} : \mathbf{grad}\;\mathbf{v}\,dx \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega), \tag{1.7}$$
$$b(\mathbf{v}, q) = -\int_\Omega q \operatorname{div}\mathbf{v}\,dx \quad \text{for all } \mathbf{v} \in \mathbf{H}^1(\Omega),\; q \in L^2(\Omega), \tag{1.8}$$

and the trilinear form

$$d(\mathbf{w};\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{w}\cdot\nabla\mathbf{u}\cdot\mathbf{v}\,dx \quad \text{for all } \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega). \tag{1.9}$$

Here

$$\mathbf{grad}\;\mathbf{u} : \mathbf{grad}\;\mathbf{v} := \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j},$$

and

$$\mathbf{w}\cdot\nabla\mathbf{u}\cdot\mathbf{v} = \sum_{i,j=1}^{2} w_j \frac{\partial u_i}{\partial x_j} v_i.$$

The weak formulation of (1.1)-(1.3) is the following: Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, we find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} a(\mathbf{u};\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= (\mathbf{f},\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\mathbf{u},q) &= 0 \quad \text{for all } q \in L_0^2(\Omega), \end{aligned} \tag{1.10}$$

where

$$a(\mathbf{u};\mathbf{u},\mathbf{v}) = a_0(\mathbf{u},\mathbf{v}) + d(\mathbf{u};\mathbf{u},\mathbf{v})$$

and $H^{-1}(\Omega)$ denotes the dual space consisting of bounded linear functionals on $H_0^1(\Omega)$.

The following theorem shows the existence of solutions (1.10) which is proved in [10].

Theorem 1.1. (see [10]) *Let $N \leq 3$ and let Ω be a bounded domain of R^N with a Lipschitz-continuous boundary Γ. Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, there exists at least one pair $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ solution of* (1.10) *or equivalently solution of* (1.1)-(1.3).

Now, we consider the uniqueness of the solution $(\mathbf{u}, p)$ of the problem (1.10). For this, we define:

$$\mathcal{N} = \sup_{\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}_0^1} \frac{d(\mathbf{w}; \mathbf{u}, \mathbf{v})}{|\mathbf{u}|_1 |\mathbf{v}|_1 |\mathbf{w}|_1}, \tag{1.11}$$

and

$$\|\mathbf{f}\|_{\mathbf{H}^{-1}} = \sup_{\mathbf{v} \in \mathbf{H}_0^1} \frac{(\mathbf{f}, \mathbf{v})}{|\mathbf{v}|_1}. \tag{1.12}$$

Theorem 1.2. *Under the hypothesis of* Theorem 1.1 *and*

$$(\mathcal{N}/\nu^2)\|\mathbf{f}\|_{\mathbf{H}^{-1}} < 1, \tag{1.13}$$

Problem (1.10) *has a unique solution $(\mathbf{u}, p)$ in $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$.*

The general theory for the finite element and mixed finite element methods for the Navier-Stokes equations are well documented in [1], [2], [3], [10], [11], [12], [14].

In this paper, we propose a new finite element method to solve the Navier-Stokes equations which is a variant of covolume scheme. For this purpose, we use the following conservative form of the Navier-Stokes equations. If we use the formula

$$\nabla \cdot (\mathbf{u}\mathbf{u}) = \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u}(\nabla \cdot \mathbf{u})$$

and divergence free condition (1.2), then (1.1) can be rewritten as

$$-\nu\, \Delta \mathbf{u} + \nabla \cdot (\mathbf{u}\mathbf{u}) + \mathbf{grad}\, p = \mathbf{f} \qquad \text{in } \ \Omega, \tag{1.14}$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \ \Omega, \tag{1.15}$$

$$\mathbf{u} = 0 \qquad \text{on } \ \Gamma. \tag{1.16}$$

For the variational formulation, we define the trilinear form:

$$d^*(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \cdot (\mathbf{u}^2) \cdot \mathbf{v}\, dx \quad \text{for all } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega). \tag{1.17}$$

Then the weak formulation is: for given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, we find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and

$p \in L_0^2(\Omega)$ such that

$$\begin{aligned} A(\mathbf{u};\mathbf{u},\mathbf{v}) - (p, \operatorname{div}\,\mathbf{v}) &= (\mathbf{f},\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ (q, \operatorname{div}\,\mathbf{u}) &= 0 \quad \text{for all } q \in L_0^2(\Omega), \end{aligned} \tag{1.18}$$

where

$$A(\mathbf{u};\mathbf{u},\mathbf{v}) = a_0(\mathbf{u},\mathbf{v}) + d^*(\mathbf{u};\mathbf{u},\mathbf{v}).$$

Since (1.18) is equivalent to (1.10) with $A(\mathbf{u};\mathbf{u},\mathbf{v}) = a(\mathbf{u};\mathbf{u},\mathbf{v})$, the existence and uniqueness theorems, i.e. Theorem 1.1, 1.2, hold for (1.18).

2. A Covolume Formulation for the Navier-Stokes Equations

We now describe a covolume method for the stationary Navier-Stokes equations. One advantage of the covolume method is that the discrete equations are derived based on local conservation of mass, momentum or energy over control volume. In [5], Chou first considered a covolume method for the Stokes problem. A MAC-like covolume method for the Stokes problem was proposed by Chou and Kwak in [6].

For covolume method, we need to define two partitions of the problem domain, which are called the primal and dual partition, respectively. Test functions are piecewise constant on the dual grid.

Let $T_h = \bigcup K_B$ be a partition of the domain Ω into a union of triangular elements, where K_B stands for the triangle whose barycenter is B. The nodes of an element are the midpoints of its sides. Let N be the number of nodal points. We denote by $P_1, P_2, \ldots, P_{N_S}$ those nodes belonging to the interior of Ω and $P_{N_{S+1}}, ..., P_N$ those on the boundary.

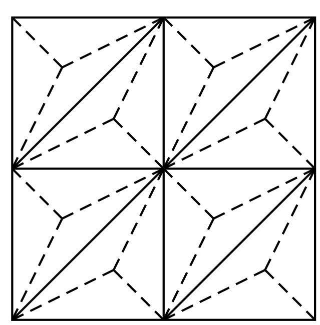

Figure 1: Primal and dual elements

The finite element space for the velocity ${\mathbf{H}_0}^h$ is the Crouzeix-Raviart space for triangles or nonconforming P_1 element [8]:

$$\begin{aligned}\mathbf{H}_0^h = \{\mathbf{v}_h \in (L^2(\Omega))^2 : \mathbf{v}_h|_{K_B} \in (P_1(K_B))^2, \quad \forall K_B \in T_h,\\ \mathbf{v}_h \text{ are continuous at the midpoints of the triangle edges and}\\ \mathbf{v}_h = 0 \text{ at the midpoints of the edges on } \partial\Omega\},\end{aligned}$$

where $P_1(K_B)$ denotes the piecewise linear function on the triangle K_B and the finite space for the pressure L_0^h is:

$$L_0^h = \{q_h \in L_0^2(\Omega) : q_h|_{K_B} \text{ is constant}, \quad \forall K_B \in T_h\}.$$

Similar nonconforming space for rectangular grid is introduced in [13] for which a parallel covolume method can be described as in [7].

Since $\mathbf{H}_0^h$ is nonconforming, the gradient and divergence operator on it must be defined piecewise:

$$(\nabla_h \mathbf{v}_h)|_{K_B} := \nabla(\mathbf{v}_h|_{K_B}),$$
$$(\text{div}_h\ \mathbf{v}_h)|_{K_B} := \text{div}\ (\mathbf{v}_h|_{K_B}).$$

On the space $\mathbf{H}_0^h$ we define the mesh dependent norms:

$$\|\mathbf{v}\|_{1,h}^2 = \sum_{i=0}^{1} |\mathbf{v}_h|_{i,K_B}^2 \quad \text{and} \quad |\mathbf{v}_h|_{i,K_B}^2 = \sum_{K_B \in T_h} \int_{K_B} |\partial_i \mathbf{v}_h|^2$$

which are also called broken norms.

Below we shall use ∇ for ∇_h and div for div_h for our convenience when there is no confusion.

Next we construct the dual partition T_h^* and the test function space. The dual grid is a union of interior quadrilaterals and border triangles.

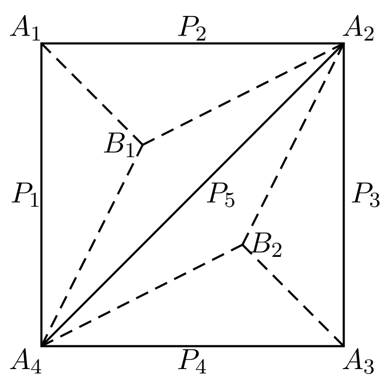

Figure 2:

For example, referring to Figure 2, the interior node P_5 belongs to the common side of the triangles $K_{B_1} = \triangle A_1A_2A_4$ and $K_{B_2} = \triangle A_2A_3A_4$ and the quadrilaterals $B_1A_2B_2A_4$ is the dual element with node at P_5. For a boundary node like P_3 the associated dual element is a triangle $\triangle A_2A_3B_2$.

We shall denote the dual partition as $T_h^* = \bigcup K_p^*$ and associate with it the test function space $\mathbf{Y}_h$, the space of certain piecewise constant vector functions. That is

$$\begin{aligned}\mathbf{Y}_0^h =&\{q \in (L^2(\Omega))^2 : q|_{K_p^*} \text{ is a constant vector, and} \\ &q|_{K_p^*} = 0 \text{ on any boundary dual element } K_p^*\}.\end{aligned}$$

Denote by χ_j^* the scalar characteristic function associated with the dual element $K_{P_j}^*, j = 1, 2, ..., N_S$. We see that for any $\mathbf{v}_h \in \mathbf{Y}_0^h$

$$\mathbf{v}_h(x) = \sum_{j=1}^{N_s} \mathbf{v}_h(P_j)\chi_j^*(x) \qquad \forall x \in \Omega. \tag{2.1}$$

As for the approximate pressure space $L_0^h \subset L_0^2(\Omega)$, we define it to be the set of all piecewise constants with respect to the primal partition since in the covolume method the pressure is assigned at the centers of triangular elements.

Finally, our test and trial function spaces should reflect the fact that in the covolume method the momentum equation (1.14) is integrated over the dual element and the continuity equation (1.15) over the primal element.

Let N_T be the number of triangles in the partition T_h. For $\mathbf{u}_h \in \mathbf{H}_0^h, \mathbf{v}_h \in \mathbf{Y}_0^h, p_h, q_h \in L_0^h$ and $\mathbf{f} \in \mathbf{H}^{-1}$, define the following trilinear form:

$$\begin{aligned}d^*(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) : &= \int_\Omega \nabla \cdot (\mathbf{u}_h^2) \cdot \mathbf{v}_h \; dx \\ &= \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \int_{\partial K_{P_i}^*} \mathbf{u}_h^2 \cdot \mathbf{n} \; ds,\end{aligned} \tag{2.2}$$

bilinear forms:

$$a_0^*(\mathbf{u}_h, \mathbf{v}_h) := -\nu \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \, ds, \tag{2.3}$$

$$b^*(\mathbf{v}_h, p_h) := \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \int_{\partial K_{P_i}^*} p_h \mathbf{n} \; ds, \tag{2.4}$$

$$c^*(\mathbf{u}_h, q_h) := -\sum_{k=1}^{N_T} q_h(B_k) \int_{K_{B_k}} div \; \mathbf{u}_h dx, \tag{2.5}$$

and

$$(\mathbf{f}, v_h) := \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \int_{K^*_{P_i}} \mathbf{f}\ dx. \tag{2.6}$$

Equation (2.2) and (2.3) are obtained by integrating the second and first terms of (1.14) against test functions, respectively and then using the second Green's identity.

Then the approximate formulation for (1.14)-(1.16) is: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_0^h \times L_0^h$ such that

$$\begin{aligned} a^*(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b^*(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h),\ \forall \mathbf{v}_h \in \mathbf{Y}_0^h\,, \\ c^*(\mathbf{u}_h, q_h) = 0, \quad \forall q_h \in L_0^h\,, \end{aligned} \tag{2.7}$$

where

$$a^*(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) = a_0^*(\mathbf{u}_h, \mathbf{v}_h) + d^*(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h).$$

Define one to one transfer operator γ_h from $\mathbf{H}_0^h$ onto $\mathbf{Y}_0^h$ by

$$\gamma_h \mathbf{v}_h = \sum_{j=1}^{N_S} \mathbf{v}_h(P_j) \chi_j^*(x) \qquad \forall x \in \Omega \tag{2.8}$$

for all $\mathbf{v}_h \in \mathbf{H}_0^h$. Using the transfer operator γ_h, we redefine the bilinear and trilinear forms in (2.7) as follows. For all $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{H}_0^h$ and $q_h \in L_0^h$

$$A_0(\mathbf{u}_h, \mathbf{v}_h) := a_0^*(\mathbf{u}_h, \gamma_h \mathbf{v}_h)\,, \tag{2.9}$$

$$D(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) := d^*(\mathbf{u}_h; \mathbf{u}_h, \gamma_h \mathbf{v}_h) \tag{2.10}$$

$$B(\mathbf{v}_h, q_h) := b^*(\gamma_h \mathbf{v}_h, q_h), \tag{2.11}$$

and

$$A(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) := A_0(\mathbf{u}_h, \mathbf{v}_h) + D(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h). \tag{2.12}$$

It is shown in [5] that the bilinear form A_0 is symmetric and that the two bilinear forms B and c^* are identical. Hence the approximation problem (2.7) becomes: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_0^h \times L_0^h$ such that

$$\begin{aligned} A(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) = (\mathbf{f}, \gamma_h \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_0^h \\ B(\mathbf{u}_h, q_h) = 0, \quad \forall q_h \in L_0^h. \end{aligned} \tag{2.13}$$

Since the redefined forms are defined only on the spaces $\mathbf{H}_0^h$ and L_0^h, reformulations help one to analyze the scheme using finite element techniques.

Chou [5] proved the following theorem for the covolume approximation for

Stokes equations:

$$\begin{aligned} A_0(\mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) &= (\mathbf{f}, \gamma_h \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_0^h \\ B(\mathbf{u}_h, q_h) &= 0, \quad \forall q_h \in L_0^h. \end{aligned} \tag{2.14}$$

Theorem 2.1. (see [5]) *Let the triangulation family of the domain* Ω *be quasi-uniform, let* $(\mathbf{u}_h, p_h)$ *be the solution of the problem* (2.14)*, and* $(\mathbf{u}, p)$ *solve the problem* (1.4)-(1.6)*. Then there exists a positive constant* C *independent of* h *such that*

$$|\mathbf{u} - \mathbf{u}_h|_{1,h} + \|p - p_h\|_0 \le Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1), \tag{2.15}$$

provided that $\mathbf{u} \in H_0^1(\Omega) \bigcap \mathbf{H}^2(\Omega), p \in H^1(\Omega)$*. Furthermore,*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1). \tag{2.16}$$

3. Convergence of the Covolume Approximation

To analyze the covolume scheme for the Navier-Stokes equations, we need some known frameworks that handle FEM for the Navier-Stokes equations [4], [9], [10].

Let X and Y be two Banach spaces and Λ a compact interval of the real line $\mathbb{R}$. We set the following class of problems:

$$F(\lambda, u) = u + TG(\lambda, u), \tag{3.1}$$

where $T \in \mathcal{L}(Y; X)$, G is $\mathcal{C}^2$ mapping from $\Lambda \times X$ into Y.

We want to find pairs $(\lambda, u) \in \Lambda \times X$ solutions of

$$F(\lambda, u) = 0. \tag{3.2}$$

We shall assume that there exists a compact interval $\Lambda \subset R$ and a branch $\{(\lambda, u(\lambda)); \lambda \in \Lambda\}$ of nonsingular solutions of (3.2) which means that $\lambda \to u(\lambda)$ is continuous function from Λ into X and

$$F(\lambda, u(\lambda)) = 0. \tag{3.3}$$

Moreover, we assume that these solutions are nonsingular in the sense that:

$$D_u F(\lambda, u(\lambda)) \text{ is an isomorphism of } X \text{ for all } \lambda \in \Lambda. \tag{3.4}$$

Now we study the approximation of the branch of nonsingular solutions. For each value of real parameter $h > 0$ which will tend to zero, we are given a finite dimensional subspace X_h of the space X and an operator $T_h \in \mathcal{L}(Y; X_h)$ intended to approximate T. We set:

$$F_h(\lambda, u_h) = u_h + T_h G(\lambda, u_h), \qquad \lambda \in \lambda, u_h \in X_h. \tag{3.5}$$

Then, the approximate problem consists in finding pairs $(\lambda, u_h) \in \Lambda \times X_h$, solutions of

$$F_h(\lambda, u_h) = 0. \tag{3.6}$$

The following theorem shows the sufficient conditions ensuring the existence and uniqueness of a branch $(\lambda, u_h(\lambda)) \in \Lambda \times X_h$ of solutions of (3.6) in a suitable neighborhood of the branch solutions of (3.3).

Theorem 3.1. (see [10]) *Assume that G is a C^2 mapping from $\Lambda \times X$ into Y and the mapping D^2G is bounded on all bounded subsets of $\Lambda \times X$. Assume in addition that the following conditions hold:*

1. There exists another Banach space Z contained in Y, with continuous imbedding, such that

$$D_uG(\lambda, u) \in \mathcal{L}(X; Z) \quad \forall \lambda \in \Lambda, \quad \forall u \in X. \tag{3.7}$$

2. We assume that

$$\lim_{h \to 0} \|(T_h - T)g\|_X = 0 \quad \forall g \in Y \tag{3.8}$$

and

$$\lim_{h \to 0} \|T_h - T\|_{\mathcal{L}(Z;X)} = 0. \tag{3.9}$$

Let $(\lambda, u(\lambda)); \lambda \in \Lambda$ be a branch of nonsingular solutions of (3.3). Then there exists a neighborhood $\mathcal{O}$ of the origin in X and for $h \leq h_0$ small enough a unique C^2 function $\lambda \in \Lambda \to u_h(\lambda) \in X$ such that:

$$(\lambda, u_h(\lambda)); \lambda \in \Lambda \text{ is a branch of nonsingular solutions of } (3.6), \tag{3.10}$$

$$u_h(\lambda) - u(\lambda) \in \mathcal{O} \quad \forall \lambda \in \Lambda. \tag{3.11}$$

Furthermore, there exists a constant $K > 0$ independent of h and λ with:

$$\|u_h(\lambda) - u(\lambda)\|_X \leq K \|(T_h - T)G(\lambda, u(\lambda))\|_X \quad \forall \lambda \in \Lambda. \tag{3.12}$$

Let us define

$$X_h := \mathbf{H}_0^h \times L_0^h,$$

and a Banach space $\tilde{X}$ as:

$$\tilde{X} := X \oplus X_h. \tag{3.13}$$

Although Theorem 3.1 is originally stated for the Navier-Stokes equations when

$$X = \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega), \quad Y = \mathbf{H}^{-1}(\Omega), \tag{3.14}$$

and

$$\mathbf{H}_0^h \subset \mathbf{H}_0^1(\Omega), \quad L_0^h \subset L_0^2(\Omega),$$

it still holds when X is replaced by $\tilde{X}$ and the norm $\|\cdot\|_X$ by the broken norm which is defined in Section 2.

So we shall apply the theorem to prove our main theorem for the covolume approximation of the Navier-Stokes equations.

Recall the covolume approximation problem (2.13) for (1.18):

Find a pair $(\mathbf{u}_h, p_h) \in \mathbf{H}_0^h \times L_0^h$ solution of

$$\begin{aligned} A(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \gamma \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_0^h, \\ (q_h, \ \operatorname{div} \mathbf{u}_h) = 0 \quad \forall q_h \in L_0^h. \end{aligned} \tag{3.15}$$

In order to study (3.15) we relate the continuous and discrete spaces by the following hypotheses:

Hypothesis H1. (Approximation Property of $\mathbf{H}_0^h$) There exists an operator $r_h \in \mathcal{L}([H^2(\Omega) \bigcap H_0^1(\Omega)]^2; \mathbf{H}_0^h)$ such that:

$$\|\mathbf{v} - r_h \mathbf{v}\|_1 \leq Ch\|\mathbf{v}\|_2 \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega). \tag{3.16}$$

Hypothesis H2. (Approximation Property of L_0^h) There exists an operator $S_h \in \mathcal{L}(L^2(\Omega); L_0^h)$ such that:

$$\|q - S_h q\|_0 \leq Ch\|q\|_1 \quad \forall q \in H^1(\Omega). \tag{3.17}$$

Hypothesis H3. (Uniform Inf-Sup Condition) For each $q_h \in L_0^h$ there exists a $\mathbf{v}_h \in \mathbf{H}_0^h$ such that:

$$(q_h, \mathrm{div}\mathbf{v}_\mathrm{h}) = \|\mathrm{q_h}\|_0^2, \quad |\mathbf{v}_\mathrm{h}|_1 \leq \mathrm{C}\|\mathrm{q_h}\|_0, \tag{3.18}$$

with a constant $C > 0$ independent of h, q_h and $\mathbf{v}_h$.

Then we have the following result:

Theorem 3.2. *Assume that the hypotheses* $\mathbf{H1}, \mathbf{H2}$ *and* $\mathbf{H3}$ *hold. Let*

$$\{(\lambda, \mathbf{u}(\lambda), \lambda p(\lambda)); \lambda = 1/\nu \in \Lambda\}$$

be a branch of nonsingular solutions of the Navier-Stokes (1.18). *Then there exists a neighborhood* $\mathcal{O}$ *of the origin in* $\tilde{X}$ *and for* $h \leq h_0$ *sufficiently small a unique* C^∞ *branch* $\{(\lambda, \mathbf{u}_h(\lambda), \lambda p_h(\lambda)); \lambda = 1/\nu \in \Lambda\}$ *of nonsingular solutions of* (3.15) *such that:*

$$(\mathbf{u}_h(\lambda), \lambda p_h(\lambda)) \in (\mathbf{u}(\lambda), \lambda p(\lambda)) + \mathcal{O} \quad \forall \lambda \in \Lambda.$$

Moreover, we have the convergence property:

$$\lim_{h \to 0} \sup_{\lambda \in \Lambda} \{|\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda)|_1 + \|p_h(\lambda) - p(\lambda)\|_0\} = 0. \tag{3.19}$$

In addition, if the mapping $\lambda \to (\mathbf{u}(\lambda), p(\lambda))$ *is continuous from* Λ *into* $\mathbf{H}^2(\Omega) \times H^1(\Omega)$, *we have for all* $\lambda \in \Lambda$*:*

$$|\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda)|_1 + \|p_h(\lambda) - p(\lambda)\|_0 \leq Kh. \tag{3.20}$$

Proof. Before applying Theorem 3.1, let us check all the conditions of the theorem. In order to do this, we recall:

$$\tilde{X} := X \oplus X_h \ \text{ and } Y = \mathbf{H}^{-1}(\Omega), \tag{3.21}$$

and define a linear operator $T \in \mathcal{L}(Y; \tilde{X})$ as follows: for given $\mathbf{f} \in Y$, $T\mathbf{f} = (\mathbf{u}_s, p_s) \in \tilde{X}$ is the solution of the Stokes problem:

$$\begin{aligned} -\nu \Delta\, \mathbf{u}_s + \mathbf{grad}\, p_s = \mathbf{f} \quad &\text{in } \Omega\,, \\ \operatorname{div} \mathbf{u}_s = 0 \quad &\text{in } \Omega\,, \\ \mathbf{u}_s = 0 \quad &\text{on } \Gamma. \end{aligned} \tag{3.22}$$

Since we are using the divergence form of the Navier-Stokes equations, define the $\mathcal{C}^2$ mapping $\tilde{G} : R_+ \times \tilde{X} \to Y$ by

$$\tilde{G}(\lambda, u) = \lambda(\nabla \cdot \mathbf{u}^2 - \mathbf{f}), \quad u = (\mathbf{u}, p) \in \tilde{X}, \tag{3.23}$$

and we see that

$$D_u\tilde{G}(\lambda, u) \cdot w = \lambda(2\nabla\mathbf{u} \cdot \mathbf{w}), \quad w = (\mathbf{w}, r) \in \tilde{X}. \tag{3.24}$$

Note that these forms are different from those defined in [10].

Let us define the space Z. By the fundamental Sobolev Imbedding Theorem, the imbedding of H_0^1 into $L^p(\Omega)$ is compact for $p < 6$. Therefore for $\mathbf{u}$ and $\mathbf{w}$ in $\mathbf{H}_0^1$, we have

$$\nabla\mathbf{u} \cdot \mathbf{w} \in (L^{3/2}(\Omega))^2.$$

So, we can choose

$$Z = (L^{3/2}(\Omega))^2 \hookrightarrow Y$$

with a compact imbedding which satisfies (3.7).

Now, let $T_h \in \mathcal{L}(Y; X_h)$ be the approximate linear operator defined by: for given $\mathbf{f} \in Y$, $(\mathbf{u}_{s,h}, p_{s,h}) = T_h\mathbf{f} \in X_h$ is the solution of

$$\begin{aligned} \nu(\nabla\mathbf{u}_{s,h}, \nabla\mathbf{v}_h) - (p_{s,h}, \operatorname{div}\, \mathbf{v}_h) = (\mathbf{f}, \gamma v_h), \quad &\forall \mathbf{v}_h \in \mathbf{H}_0^h, \\ (r_h, \operatorname{div}\, \mathbf{u}_h) = 0, \quad &\forall r_h \in L_0^h. \end{aligned} \tag{3.25}$$

As it is proved for the Stokes problem in [10], under the hypotheses **H1**, **H2** and **H**3,

$$\lim_{h\to 0} \{|\mathbf{u}_{s,h} - \mathbf{u}_s|_1 + \|p_{s,h} - p_s\|_0\} = 0, \tag{3.26}$$

i.e.

$$\lim_{h\to 0} \|(T_h - T)\mathbf{f}\|_{\tilde{X}} = 0, \quad \forall\, \mathbf{f} \in Y.$$

Moreover, when $(\mathbf{u}_s, p_s)$ belongs to $\mathbf{H}^2(\Omega) \times H^1(\Omega)$ we have the error bound by the covolume analysis for the Stokes case [5]:

$$|\mathbf{u}_{s,h} - \mathbf{u}_s|_1 + \|p_{s,h} - p_s\|_0 \le Ch(\|\mathbf{u}_s\|_2 + \|p_s\|_1), \tag{3.27}$$

i.e

$$\|(T_h - T)\mathbf{f}\|_{\tilde{X}} \le Ch\|T\mathbf{f}\|_{\mathbf{H}^2(\Omega)\times H^1(\Omega)}.$$

Therefore the compactness of the imbedding of Z into Y together with (3.26) imply that

$$\lim_{h\to 0} \|(T_h - T)\|_{\mathcal{L}(Z;\tilde{X})} = 0.$$

Thus (3.8) and (3.9) hold. Since

$$A(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) = \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + \left(\nabla \cdot \mathbf{u}_h^2, \gamma \mathbf{v}_h\right)$$

for the Navier-Stokes equations, (3.15) can be expressed as follows:

$$(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (1/\nu)(p_h, \mathrm{div}\ \mathbf{v}_h) = (1/\nu)\left(\mathbf{f} - \nabla \cdot \mathbf{u}_h^2, \gamma \mathbf{v}_h\right), \qquad \forall \mathbf{v}_h \in \mathbf{H}_0^h,$$

$$(q_h, \mathrm{div}\ \mathbf{u}_h) = 0 \quad \forall q_h \in L_0^h.$$

By (3.25), $u_h := (\mathbf{u}_h, (1/\nu)p_h)$ satisfies:

$$u_h = T_h\left[(1/\nu)\left(\mathbf{f} - \nabla \cdot \mathbf{u}_h^2\right)\right] = -T_h\tilde{G}(1/\nu, u_h).$$

Thus, an equivalent form of problem (3.15) is: find $u_h \in X_h$ solution of

$$F_h(\lambda, u_h) = u_h + T_h\tilde{G}(\lambda, u_h) = 0 \quad \text{with } \lambda = 1/\nu.$$

As a consequence, we can apply the conclusion of Theorem 3.1: for $h \le h_0$ sufficiently small there exists a unique branch $\{(\lambda, u_h(\lambda) = (\mathbf{u}_h(\lambda), \lambda p_h(\lambda))); \lambda \in \Lambda\}$ of nonsingular solutions of (3.15) which is equivalent to the equation

$$u_h + T_h\tilde{G}(\lambda, u_h) = 0, \quad \forall \lambda \in \Lambda,$$

and a real number $a > 0$, independent of h, such that:

$$\|u_h(\lambda) - u(\lambda)\|_{\tilde{X}} \le a \quad \forall \lambda \in \Lambda.$$

Furthermore, (3.12) implies that

$$|u_h(\lambda) - u(\lambda)|_1 + |\lambda|\|p_h(\lambda) - p(\lambda)\|_0 \le K\|(T_h - T)\tilde{G}(\lambda, u(\lambda))\|_{\tilde{X}}.$$

Hence (3.19) follows from (3.26). Since

$$u(\lambda) = (\mathbf{u}(\lambda), \lambda p(\lambda)) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$$

is the solution of the Stokes system:

$$u(\lambda) = -T\tilde{G}(\lambda, u(\lambda)),$$

the error estimate for covolume scheme with $\tilde{G}(\lambda, u_h(\lambda))$ as right hand side gives:

$$\|(T_h - T)\tilde{G}(\lambda, u(\lambda))\|_{\tilde{X}} \leq Ch\{\|\mathbf{u}(\lambda)\|_2 + \|p(\lambda)\|_1\}.$$

Thus (3.20) is satisfied by the continuity of the mapping $\lambda \to u(\lambda)$ from Λ into $\mathbf{H}^2(\Omega) \times H^(\Omega)$. □

Remark 1. Although we used the divergence form of the Navier-Stokes equations for the covolume method, the original form with $\mathbf{u} \cdot \nabla \mathbf{u}$ can also be used to derive another covolume scheme and all the results obtained here hold similarly.

4. Numerical Results

The discrete system resulting from a covolume scheme for the Navier-Stokes equations (2.13) constitutes the following nonlinear system of algebraic equations:

$$\begin{aligned} \mathbf{A}\mathbf{u} + \mathbf{C}(\mathbf{u}) + \mathbf{B}\mathbf{p} &= \mathbf{F}\,, \\ \mathbf{B}^T\mathbf{u} &= 0\,, \end{aligned} \tag{4.1}$$

where

$$(\mathbf{u})^T = [\mathbf{u}_1^T \;\; \mathbf{u}_2^T], \quad \mathbf{u}_i^T = [u_i^1 \;\cdots\; u_i^N], \quad i = 1, 2,$$

for N nodal velocity and

$$\mathbf{p}^T = [p_1 \;\cdots\; p_L],$$

where L is the number of elements in discretization.

Matrices $\mathbf{A}$ and $\mathbf{B}$ are $2N \times 2N$ and $2N \times L$, respectively. The force vector $\mathbf{F}$ is $2N \times 1$ and $\mathbf{C}(u)$ is nonlinear term. Just for simplicity, we assume $\nu = 1$.

The literatures on incompressible flows contain a variety of different methods of solving such nonlinear problems, which are all based on a different linearization of the system. In this paper, we consider a simple Picard type iteration method which is used to solve (4.1) as a sequence of linear problems for $\mathbf{u}_k, \mathbf{p}_k$ at iterate k. Given $(\mathbf{u}_0, \mathbf{p}_0)$, for $k = 1, 2, \ldots$ solve

$$\begin{aligned} \mathbf{A}\mathbf{u}_k + \mathbf{B}\mathbf{p}_k &= \mathbf{F}_k - \mathbf{C}(\mathbf{u}_{k-1}) \\ \mathbf{B}^T\mathbf{u}_k &= 0 \end{aligned} \tag{4.2}$$

and we have essentially a sequence of Stokes type problems in which the force vector on the right is iteratively adjusted to accommodate the convective nonlinear term. To solve system (4.2) in each iteration we use the Uzawa algorithm with Conjugate directions.

Example. We have chosen the following test problem on the unit rectangular domain $\bar{\Omega} = [0,1] \times [0,1]$ with exact solution

$$u_1(x,y) = x^2(x-1)^2 y(y-1)(2y-1)\,,$$
$$u_2(x,y) = -x(x-1)(2x-1)y^2(y-1)^2\,,$$
$$p(x,y) = 2(x-\frac{1}{2})(y-\frac{1}{2})\,.$$

We compare with FEM solutions. The results are shown in Table.1.

Comparing the nonconforming covolume method with the nonconforming finite element method, we can conclude that the numerical results are almost the same but the velocity error with the covolume method is marginally better than the finite element method.

	Nonconforming covolume				Nonconforming FEM			
	$\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}$		$\|p-p_h\|_{L_2(\Omega)}$		$\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}$		$\|p-p_h\|_{L_2(\Omega)}$	
n	error	order	error	order	error	order	error	order
8	1.428e-03		3.261e-02		1.429e-03		3.257e-02	
16	4.416e-04	1.69	1.441e-02	1.18	4.423e-04	1.69	1.441e-02	1.18
32	1.190e-04	1.89	6.536e-03	1.14	1.192e-04	1.89	6.532e-03	1.14
64	3.050e-05	1.96	3.112e-03	1.07	3.057e-05	1.96	3.109e-03	1.07
128	7.68e-06	1.99	1.529e-03	1.03	7.70e-06	1.99	1.524e-03	1.03

Table 1: Errors and orders of convergence for the triangular meshes

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