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A GENERALIZATION OF RIESZ POTENTIALS AND FRACTIONAL POWERS OF DIFFERENTIAL OPERATORS

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Abstract: In this short paper an operator obtained by convolution with a distributional function wich belongs to the $\{W_\alpha(P, m, n)\}_{\alpha\in\mathbb{C}}$ familie is presented. From it as particular cases one may obtain both, elliptic or hyperbolic Riesz potentials. Certain elementary properties are studied and the inversion problem is presented using, as principal tool, the Laplace transform.

Moreover we obtain an expression that may be interpreted as a fractional power of the ultrahyperbolic differential operator.

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1. Introduction

The theory of Riesz potentials have been studied widely by M. Riesz (cf. [5]), Samko (cf. [7]), Nogin (cf. [3]), Rubin (cf. [6]).

For a function f belongs to S, the Schwartzian space of functions, the Riesz potential of order α, α a complex number that $Re\,(\alpha) > 0$, is defined as

$$(R^\alpha f)\,(x) = \frac{1}{H_n(\alpha)} \int \frac{f(y)\,dy}{|x-y|^{n-\alpha}}\,, \tag{1.1}$$

where the constant $H_n(\alpha)$ is given by

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$$H_n(\alpha) = \frac{2^{\alpha}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}\,. \tag{1.2}$$

The operator R^{α} is also written as a convolution of the density f with the Riesz elliptic kernel

$$R_{\alpha}(|x|) = \frac{|x|^{\alpha-n}}{H_n(\alpha)}\,, \tag{1.3}$$

where $|x| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}$.

Analogously, by the convolution with the ultrahyperbolic Riesz kernel (2.3) it is obtained the ultrahyperbolic Riesz potentials.

Both, the elliptic and the ultrahyperbolic Riesz potentials have the semigroup property.

The inverse of this potentials are, formally, fractional powers of the Laplacian and the ultrahyperbolic differential operators (cf [7], also [8])

In this paper, an operator that may be considered as a generalization of the ultrahyperbolic Riesz potential is introduced. It is defined as the convolution $W_{\alpha}(P,m,n) * f$ where $W_{\alpha}(P,m,n)$ is given by (1.1) and f is a function belongs to the $\mathcal{C}_0^{\infty}(K_+)$ the class of all $\mathcal{C}^{\infty}$ functions on $\mathbb{R}^n$ with compact support included in the cone K_+, the light cone. We give some elementary properties and we also establish a connection between the $W^{\alpha}f$ operator and certain differential operators.

The paper is organized as follows. In Section 2 we present the notation and the elementary previous results which will be used in the following section. Also the operator $W^{\alpha}f$, that generalize the classical Riesz potential, is introduced. In Section 3, Theorem 1 proves the semigroup property and Theorem 2 shows the relation between the operator W^{α} and the Klein-Gordon differential operator iterated l times. As particular cases we obtain the connections with the ultrahyperbolic differential operator and with the Laplacian.

Finally by using the Laplace transform the inverse operator $(W^{\alpha})^{-1}$ is obtained.

2. Preliminaries

As usual $t = (t_1, t_2, \ldots, t_n)$ denote a point of the n-dimensional Euclidean space R^n. Let $P = P(t)$ be the quadratic nondegenerate form in n variables of the form

$$P(t) = t_1^2 + \cdots + t_p^2 - t_{p+1}^2 - \ldots - t_{p+q}^2\,, \tag{2.1}$$

where $p+q=n$, n the dimension of the space.

By K_+ we denote the cone $K_+=\{t\in R^n : t_1>0, P>0\}$ and by $\overline{K}_+$ its closure. Let $z=(z_1,z_2,\cdots,z_n)$ be a point of C^n, $z_\gamma=x_\gamma+iy_\gamma$, $\gamma=1,2,\ldots,n$, $\langle t,z\rangle = t_1z_1+t_2z_2,\ldots+t_nz_n$, and $dt=dt_1.dt_2\ldots dt_n$. The tube T_- is by definition $T_-=\{z\in C^n : y\in V_-\}$, where $V_-=\{y\in R^n : y<0, P(y)>0\}$ and the tube $T_+=\{z\in C^n : y\in V_+\}$, where $V_+=\{y\in R^n : y_1>0, P(y)>0\}$.

In the next we will consider two families of functions supported on the light cone given by the following definitions.

Definition 1. Let α be a complex parameter. Following Riesz (cf. [5], p. 89) Schwartz (cf. [9], p. 179) and Trione (cf. [10]), we consider the family of retarded functions depending on Lorentz metric

$$W_\alpha(P,m,n)=\begin{cases}\frac{(m^{-2}P)^{\frac{\alpha-n}{4}}}{\pi^{\frac{n-2}{2}}2^{\frac{\alpha+n-2}{2}}\Gamma(\frac{\alpha}{2})}J_{\frac{\alpha-n}{2}}(m^2P)^{\frac{1}{2}} & \text{if } t\in K_+\,,\\ 0 & \text{if } t\notin K_+\,,\end{cases} \tag{2.2}$$

where m is non negative real number, n the dimension of the space and $J_\gamma(z)$ is the Bessel function of first kind defined by the formula

$$J_\gamma(z)=\sum_{p=0}^{\infty}\frac{(-1)^p\left(\frac{z}{2}\right)^{\gamma+2p}}{\Gamma(\gamma+p+1)}\,.$$

It may be observed that $W_\alpha(P,m,n)$ that is an ordinary function if Re $\alpha\geq n$, is a distributional entire function of α (cf. [2])

Definition 2. From (2.2) when $m=0$ and we replace $J_{\frac{\alpha-n}{2}}$ by its Taylor series it result the ultrahyperbolic kernel due by Nozaki (cf. [4]), given by

$$\Phi_\alpha(P)=\begin{cases}\frac{\Gamma^{\alpha-n}}{C_n(\alpha)} & \text{if } t\in K_+\,,\\ 0 & \text{if } t\notin K_+\,,\end{cases} \tag{2.3}$$

where

$$\Gamma^{\alpha-n}=\left(t_1^2+...+t_p^2-t_{p+1}^2-...-t_{p+q}^2\right)^{\frac{\alpha-n}{2}};\, t_1>0;\, p+q=n$$

and

$$C_n(\alpha)=\frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2-\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}\,. \tag{2.4}$$

Definition 3. By putting $p=1$ in (2.3) we obtain immediately

$$R_\alpha(P)=\begin{cases}\frac{P^{\frac{\alpha-n}{2}}}{H_m(\alpha)} & \text{if } t\in\Gamma_+\,,\\ 0 & \text{if } t\notin\Gamma_+\,,\end{cases} \tag{2.5}$$

where

$$H_m(\alpha) = 2^{\alpha-1}\pi^{-1+\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+2-n}{2}\right);$$

$R_\alpha(P)$ is the hyperbolic kernel introduced by Riesz.

Moreover, by putting $n = 1$ in $R_\alpha(P)$, and taking into account the Legendre's duplication formula of $\Gamma(z)$:

$$\Gamma(2z) = 2^{2z-1}\pi^{\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

we get

$$I_\alpha = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{if } t > 0\,, \\ 0 & \text{if } t < 0\,. \end{cases} \tag{2.6}$$

In the next we shall deal with the Laplace transform of functions or distributions depending of Lorentz metric, then we introduce the following definition.

Definition 4. Let $f(t)$ be function having the properties.

1. $f(t)$ is a function depending on P.
2. $\text{supp} f(t) \subset \overline{K}_+$.
3. $e^{<t,y>}f(t) \in L_1$ if $y \in V_-$.

The Laplace transform of $f(t)$, is given by

$$L(f)(z) = \int_{R^n} e^{-i\langle t,z\rangle} f(t)\,dt\,.$$

From Gonzalez Dominguez and Trione (cf. [2], also [10]) we known that the Laplace transform of $W_\alpha(P,m,n)$ is

$$L\left(W_\alpha(P,m,n)\right) = \left(\rho^2+m^2\right)^{-\frac{\alpha}{2}} \tag{2.7}$$

if Re $\alpha > 2n-4$, and Re $\left(z_{p+1}^2+\cdots+z_{p+q}^2-z_1^2-\cdots-z_p^2\right)^{\frac{1}{2}} > 0$ where $\rho^2 = z_{p+1}^2+\cdots+z_{p+q}^2-z_1^2-\cdots-z_p^2$.

Since $m \neq 0$, ρ^2+m^2 never vanishes, the function in the right hand side of (2.7) is an entire function of α and by analytical in continuation the formula (2.7) is valid for all α (cf. [4]).

Furthermore because the right hand member of (2.7) is an analytic function it is deduced that the distributional function $W_\alpha(P,m,n)$ is an entire distributional function of α (cf. [2], also [10]).

Definition 5. Let φ be a function belongs to $\mathcal{C}_0^\infty$ whose support is contained in the positive light cone K_+. Then a generalized Riesz potential of order α of φ denoted $W^{\alpha}\varphi$, is defined by the convolution

$$W^{\alpha}\varphi = W_{\alpha}(P, m, n) * \varphi \tag{2.8}$$

for $\alpha \geq 2n - 4$.

From the convolution theorem for the Laplace transform it results

$$L\left(W^{\alpha}\varphi\right)(z) = \left(\rho^2 + m^2\right)^{-\frac{\alpha}{2}} L(\varphi)(z)\,. \tag{2.9}$$

The last equation yields the following representation for the operator W^{α} via the Laplace transforms L and L^{-1}, namely

$$\left(W^{\alpha}\varphi\right)(x) = L^{-1}\left(\left(\rho^2 + m^2\right)^{-\frac{\alpha}{2}} L(\varphi)(z)\right)(x)\,. \tag{2.10}$$

This last formula is a generalization of one dimensional due to Samko (cf. [8], p. 141, formula (7.17)).

In fact, if $m = 0$ and $n = 1$ is considered, from (2.10) we obtain

$$(W^{\alpha}\varphi)(x) = \left(I_0^{\alpha}\varphi\right)(x) = L^{-1}\left(z^{-\alpha}L(\varphi)(z)\right)(x)\,, \tag{2.11}$$

where the left hand member by virtue of (2.6) is the convolution

$$\left(\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * \varphi\right)(x) = \left(I_0^{\alpha}\varphi\right)(x)\,, \tag{2.12}$$

where I_0^{α} denote the Riemann-Liouville fractional integral of order α.

Let us now observe the ultrayperbolic Klein-Gordon differential operator iterated "l" times given by

$$K^l = \left(m^2 + \Box\right)^l = \left\{m^2 + \sum_{i=1}^{p} \frac{\partial^2}{\partial t_i^2} - \sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial t_i^2}\right\}^l\,, \tag{2.13}$$

where $p+q = n$, n the dimension of the space $\mathbb{R}^n$. We observe that when $m = 0$ (2.13) reduces to the ultrahyperbolic differential operator iterated l times

$$\Box^l = \left\{\sum_{i=1}^{p} \frac{\partial^2}{\partial t_i^2} - \sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial t_i^2}\right\}^l\,. \tag{2.14}$$

When the number of negative terms $q = 0$, (2.14) reduces to Laplacian iterated l times

$$\Delta^l = \sum_{i=1}^{p} \frac{\partial^2}{\partial t_i^2}\,. \tag{2.15}$$

3. Properties of the Operator W^α

On the basis of the properties of the functions $W_\alpha(P, m, n)$ we will prove some theorems related to the potentials $W^\alpha\varphi$ that we have introduced in (2.8).

Firstly we will show that the operator W^α admits the semigroups property.

We know (cf. [9], p. 177) that the convolution $W_\alpha * W_\beta$ exists for every couple α, β of complex numbers. This particular quality give us a possible way to prove the semigroup property, but we prefer do it by using the Laplace transform.

Theorem 1. *Let α and β be complex numbers with* $\operatorname{Re}\alpha > 0$ *and* $\operatorname{Re}\beta > 0$. *Then is valid the composition formulae.*

$$W^\alpha W^\beta \varphi = W^{\alpha+\beta}\varphi . \tag{3.1}$$

Proof. In fact, applying the Laplace transform in (3.1), according to (2.9) we have

$$\begin{aligned} L\left(W^\alpha W^\beta \varphi\right) &= L\left(W_\alpha * W^\beta \varphi\right) = L\left(W_\alpha\right) L\left(W^\beta \varphi\right) \\ &= \left(\rho^2 + m^2\right)^{-\frac{\alpha}{2}} L\left(W^\beta \varphi\right) \\ &= \left(\rho^2 + m^2\right)^{-\frac{\alpha}{2}} \left(\rho + m^2\right)^{-\frac{\beta}{2}} L(\varphi) \\ &= \left(\rho^2 + m^2\right)^{-\frac{\alpha+\beta}{2}} L(\varphi) \end{aligned}$$

and the Theorem 1 it results. □

Relation between the generalized ultrahyperbolic Riesz potential and the Klein-Gordon differential operator is given by the following theorem.

Theorem 2. *Let α be a positive real number, $\alpha \geq 2l$, $l = 1, 2, \cdots$. Let K be the ultrahyperbolic Klein-Gordon operator iterated l-times and $W^\alpha\varphi$ the generalized Riesz potential of order α of the function φ, φ belongs to $\mathcal{C}_0^\infty(\Gamma_+)$. Then*

$$K^l \left\{W^\alpha \varphi\right\} = W^{\alpha-2l}\varphi = W^\alpha \left\{K^l \varphi\right\} . \tag{3.2}$$

Proof. By definition, we have

$$W^{\alpha-2l}\varphi = W_{\alpha-2l} * \varphi = W_\alpha * W_{-2l} * \varphi = W_\alpha * K^l \varphi = W^\alpha \left\{K^l \varphi\right\} \tag{3.3}$$

and

$$W^{\alpha-2l}\varphi = W_{-2l+\alpha} * \varphi = W_{-2l} * W^\alpha \varphi = K^l \left\{W^\alpha \varphi\right\} . \tag{3.4}$$

Then, by virtue of (3.3) and (3.4) and the definitory formula (2.8) we have

$$K^l \{W^\alpha \varphi\} = W^\alpha \left\{K^l \varphi\right\} = W^{\alpha-2l}\varphi \tag{3.5}$$

which is the thesis. □

Taking into account the above considerations the potential defined by (2.8) contain as particular cases the ultrahyperbolic Riesz potential with the Nozaki kernel, the hyperbolic and the Euclidean Riesz potentials.

Moreover we have the following corollaries of Theorem 2.

Corollary 1. *Let α be a positive real number, $\alpha \geq 2l$, $l = 1, 2, \cdots$. Let $\Box^l$ be the ultrahyperbolic differential operator iterated l times, and $R^\alpha\varphi$ the ultrahyperbolic Riesz potential of order α. Then*

$$\Box^l\{R^\alpha \varphi\} = R^{\alpha-2l}\varphi = R^\alpha\{\Box^l\varphi\}\,. \tag{3.6}$$

Proof. From (3.5) and taking into account (2.3) and (2.14) it results. □

Corollary 2. *The same hypothesis that in Theorem 2 on α and l. Let Δ^l be the Laplacian iterated l times, and let $R_e^\alpha\varphi$ the Euclidean Riesz potential of order α. Then*

$$\Delta^l\{R_e^\alpha \varphi\} = R_e^{\alpha-2l}\varphi = R_e^\alpha\{\Delta^l\varphi\}\,. \tag{3.7}$$

Proof. The corollary follows from (3.5), (2.13), and by putting $m = 0$, $q = 0$ in (2.1).

It may be observed that (3.7) is the same that the one due to Samko (cf. [7], p. 1105]).

Also, formulae (3.6) are analogue of the formula due to us (cf. [1]) for causal Riesz potentials. □

The Inverse Operator $(W^\alpha)^{-1}$

Our main objective in this paragraph is to obtain an operator that allows us to solve the following inversion problem. Let φ be a function belongs to $\mathcal{C}_0^\infty(K_+)$. We assume that $W^\alpha\varphi = f$, for some f. How to recover φ?.

Since the operator W^α has the semigroup property is natural to expect that the inverse operator noted $(W^\alpha)^{-1}$ will be given, at least formally, by the expression

$$(W^\alpha)^{-1} f = W_{-\alpha}(P, m, n) * f \tag{3.8}$$

which must be interpreted in an appropriate sense. It is well known (cf. [8], also

[1]) that fractional powers of differential operators, i.e. the operators that are defined, by using Fourier transforms, by means of multiplication by a fractional power of the quadratic form given by (2.1) work as the inverse potentials operators. Now using the Laplace transform we obtain an expression that may be interpreted as a fractional potential of the ultrahyperbolic differential operator and an inverse $(W^{\alpha})^{-1}$ operator.

We have the following theorem.

Theorem 3. *Let α be a real positive number. If $f = W^{\alpha}\varphi$, then $(W^{\alpha})^{-1} f = \varphi$, where $(W^{\alpha})^{-1} = K^{\frac{\alpha}{2}}$, being $K^{\frac{\alpha}{2}}$ a fractional power of the Klein-Gordon differential operator.*

Proof. Firstly we consider the case $\alpha = 2l$, $l = 1, 2, \cdots$. From (3.5) we have

$$K^{l}\left\{W^{\alpha}\varphi\right\} = W_{\alpha-2l} * \varphi = W_0 * \varphi = \delta * \varphi = \varphi\,. \tag{3.9}$$

Then the inverse operator $(W^{\alpha})^{-1}$ is given by K^{l}. □

Let α be a real positive number. Because $W_{\alpha}(P, m, n)$ is an entire distributional function on α, it is a continuos function on α and then we have

$$W_{\alpha-2\left(\frac{\alpha}{2}\right)} * \varphi = \delta * \varphi = \varphi\,. \tag{3.10}$$

On the other hand from the property of the Laplace transform we get

$$L\left(\left(\Box + m^2\right)^{l}\delta\right) = \left(m^2 + \rho^2\right)^{l}\,. \tag{3.11}$$

The right hand member of (3.11) exist for all real number, then for $l = \frac{\alpha}{2}$ we have

$$L\left(\left(\Box + m^2\right)^{\frac{\alpha}{2}}\delta\right) = \left(m^2 + \rho^2\right)^{\frac{\alpha}{2}}\,. \tag{3.12}$$

Then, from (3.5) and putting $l = \frac{\alpha}{2}$ we have

$$K^{\frac{\alpha}{2}}\left\{W^{\alpha}\varphi\right\} = \varphi\,. \tag{3.13}$$

From the last equation we have

$$\left(W^{\alpha}\right)^{-1} = W_{-\alpha} = K^{\frac{\alpha}{2}} \tag{3.14}$$

which is the thesis of Theorem 3. □

Corollary 1. *If $m = 0$, it result*

$$\Box^{\frac{\alpha}{2}}\left\{R_H^{\alpha}\varphi\right\} = \varphi\,. \tag{3.15}$$

Corollary 2. *If $m = 0$ and $q = 0$, it result*

$$\Delta^{\frac{\alpha}{2}}\left\{R_e^{\alpha}\varphi\right\} = \varphi\,. \tag{3.16}$$

The formulas (3.15) and (3.16) coincides with the well known expression of the inverse of the ultrahyperbolic Riesz potential and the inverse of the elliptic

Riesz potential (cf. [7]).

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