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A SUFFICIENT CONDITION FOR CLIFFORD'S INEQUALITY FOR RANK 1 SHEAVES ON REDUCIBLE CURVES

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Abstract: Here we give some very restrictive assumptions, which imply a Clifford's Theorem and a generalization of it (Martens) for rank 1 sheaves on reducible curves.

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1. Introduction

A cursory reading of [2] and [3] shows that Clifford's Inequality for special line bundles often fails on reducible curves. Here we show that a non-degeneracy assumption allows us to adapt the classical approach ([1], [5], Theorem A of Appendix) and give the following results.

Proposition 1. *Let X be a reduced and projective curve and F a coherent sheaf on X with depth 1 and pure rank 1. Assume that F is spanned and locally free outside a finite set $S \subset X$ such that X is either smooth or with an ordinary node or with an ordinary cusp at each point of S. Assume that the image of each irreducible component of X by the morphism $\phi : X \backslash S \to \mathbb{P}^n$, $n := h^0(X, F) - 1$, induced by $H^0(X, F)$ spans $\mathbb{P}^n$. Set $S' := S \cap Sing(X)$, $s := \sharp(S)$, $s' := \sharp(S')$, $d := \deg(F)$ and $g := p_a(X)$. Assume that $X \backslash S'$ is connected and that $d \leq 2g - 2 - s'$. Then:*

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(a) $h^0(X, F) \le (d - s')/2 + 1$.

(b) We have $h^0(X, F) = (d - s')/2 + 1$ *if and only if* F *is spanned* $S = S'$ *and the following holds. Let* $u : X \to Y$ *be the partial normalization of* X *in which we only normalize the point of* $S = S'$. *Set* $L := u^*(F)/\mathrm{Tors}(u^*(F))$. *The sheaf* L *is a degree* $d - s'$ *spanned line bundle on* Y, $p_a(Y) = g - s'$, $h^0(Y, L) = h^0(X, F)$ *and either* X *is Gorenstein,* $L \cong \omega_Y$ *and* ω_Y *is very ample or the morphism* $\phi_L : Y \to \mathbb{P}^n$, $n := h^0(X, F) - 1 = (d - s')/2$, *is the composition of a degree 2 finite and flat morphism* $\psi : Y \to \mathbb{P}^1$ *with the order* $(d - s')/2$ *Veronese embedding* $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$. *In this case* X *has only planar singularities.*

Theorem 1. *Fix any reduced and Gorenstein projective curve and* $S \subseteq \mathrm{Sing}(X)$ *such that* $X \backslash S$ *is connected and every point of* S *is either an ordinary node or an ordinary cusp of* X. *Set* $g := p_a(X)$ *and* $s := \sharp(S)$. *Assume* $g - s \ge 2$ *and fix integers* d, n *such that* $n \ge 1$, $d \le 2g - 2 - s$. *Let* Γ *be an irreducible family of depth 1 sheaves* F *on* X *with pure rank 1,* $\deg(F) = d$, $\mathrm{Sing}(F) = S$, $h^0(X, F) \ge n + 1$ *and* F *spanned outside* S. *Set* $\gamma := \dim(\Gamma)$. *Assume that for every* $F \in \Gamma$ *the image of each irreducible component of* X *by the morphism* $\phi_F : X \backslash S \to \mathbb{P}^x$, $x := h^0(X, F) - 1$, *induced by* $H^0(X, F)$ *spans* $\mathbb{P}^n$. *Then* $\dim(\Gamma) \le d - 2n$.

Theorem 1 is an extension to reducible curves of a theorem of H. Martens ([1], p. 193).

2. The Proofs

Remark 1. Let $u : Y \to X$ be a partial normalization of X, i.e. let Y be a reduced projective curve (not necessary connected) and u a finite and surjective morphism such that there is an open and dense subset U of Y with $u|U : U \to u(U)$ an isomorphism. Let F be depth 1 sheaf on X with pure rank r. Set $G := u^*(F)/\mathrm{Tors}(u^*(F))$. The coherent sheaf G is a depth 1 sheaf on Y with pure rank r and there is an injective map $j_{F,u} : F \to u_*(F)$ with cokernel with finite support. Applying Riemann-Roch on Y and on X we get $\deg(G) = \deg(F) + \chi(\mathcal{O}_X) - \chi(\mathcal{O}_Y)$. Now assume that F is spanned. Since the tensor product is a right exact functor, $u^*(F)$ is spanned. Hence G is spanned. Since F and G are spanned, the natural map $H^0(X, F) \to H^0(Y, G)$ is injective.

We need the following well-known lemma whose proof is left to the reader (if each point of S is an ordinary node, then the classification of rank 1 sheaves on the local rings $\mathcal{O}_{X,P}$, $P \in S$, is given in [8], pp. 164–166; if some point of S is an ordinary cusp of X we also need to quote the classification of depth 1 modules with pure rank on any A_n-singularity given in [6]).

Lemma 1. *Let X be a reduced projective curve and F a depth 1 sheaf on X with pure rank 1. Set $S := Sing(F)$ and $s := \sharp(S)$. Assume that each point of X is either an ordinary node or an ordinary cusp. Let $u : Y \to X$ be the partial normalization of X in which we only normalize the points of S. Let F be a depth 1 sheaf on X with pure rank r. Fix $S \subseteq Sing(F)$ such that $u|Y\backslash u^{-1}(S) \to Y\backslash S$ is an isomorphism. Set $G := u^*(F)/Tors(u^*(F))$. Then:*

(a) The sheaf G has depth 1, pure rank 1 and $\deg(G) = \deg(F) - \sharp(S)$.

(b) $G \cong u^*(u_*(G))/Tors(u^*(u_*(G)))$, $F \cong u_*(G)$ and $h^i(Y,G) = h^i(X,F)$, $i = 0,1$.

Proof of Proposition 1. Let G be the image of the evaluation map $H^0(X,F)\otimes \mathcal{O}_X \to F$. G is locally free with rank 1 outside S, F/G is supported by P, $h^0(X,G) = h^0(X,F)$, and ϕ is induced by $H^0(X,G)$.

(i) Here we assume $S = \emptyset$, i.e. $G = F$ and F locally free. Since each irreducible component of $\phi(X)$ spans $\mathbb{P}^n$, the pairing

$$\psi_F : H^0(X,F) \otimes H^0(X, Hom(F,\omega_X)) \to H^0(X,\omega_X)$$

is 1-degenerate in both variables. Here we do not need that ω_X is locally free, because the pairing is well-defined by the locall freeness of F. Riemann-Roch and Serre duality gives $h^0(X,F) - h^0(X, Hom(F,\omega_X)) = d + 1 - g$. Thus (a) follows from the case $k = 1$ of [4], Proposition 1.3. Thus we get part (a) under this assumption.

(ii) Now assume $S' = \emptyset$, but $S \neq \emptyset$. Since each point of $S\backslash S'$ is a smooth point of X, G is locally free and spanned. The first part gives a contradiction.

(iii) Here we assume $S = S' \neq \emptyset$. We have $p_a(Y) = g - s$. Let $u : X \to Y$ be the partial normalization of X in which we only normalize the point of $S = S'$. Set $L := u^*(F)/\mathrm{Tors}(u^*(F))$. The sheaf L is a degree $d - s'$ spanned line bundle on Y and $h^0(Y,L) = h^0(X,F)$. Since $\deg(L) \le 2p_a(Y - 2$, we may apply (i) to L and get $h^0(Y,L) \le (d-s)/2 + 1$. Since $s = s'$ and $h^0(Y,L) = h^0(X,F)$, (a) is proved in this case.

(iv) Here we assume $S \neq S' \neq \emptyset$. Since each point of $S\backslash S'$ is a smooth point of X, we may apply step (iii) to G and get a contradiction, concluding the proof of (a).

(v) Take F such that $h^0(X,F) = (d-s')/2 + 1$. Since X has a planar singularitiy at each point of S, X has planar singularities (resp. it is Gorenstein) if and only if Y has planar singularities (resp. it is Gorenstein). We saw in (ii) and (iii) that $S = S'$ and F is spanned. Take Y, u and L as in (iii). Notice that $L \cong \phi_L^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Let $\mathcal{B}(Y)$ denote the set of the irreducible components of Y.

Since u induces a bijection $Y\backslash u^{-1}(S) \to X\backslash S$, it induces a bijection between $\mathcal{B}(Y)$ and the set $\mathcal{B}(X)$ of all irreducible components of X. The morphism $\phi_L : Y \to \mathbb{P}^n$, $n = (d-s')/2$, induced by $|L|$ has the following property:

$\diamond$ for every irreducible component T of Y, $\phi_L(T)$ spans $\mathbb{P}^n$.

In particular ϕ_L does not contract to a point any irreducible component of Y. Hence

$$d - s = \sum_{T\in\mathcal{B}(Y)} (\deg(\phi_L|T)(\deg(\phi_L(T))). \tag{1}$$

The property $\diamond$ implies $\deg(\phi_L(T)) \geq n$ for all $T \in T \in \mathcal{B}(Y)$. Hence either $\phi_L(Y)$ is irreducible or Y has two irreducible components, each of them mapped isomorphically onto a rational normal curve. If Y is irreducible, then we may apply [5], Theorem A, of Appendix, and get that $Y, L)$ is in one of two cases listed in (b). Now assume that Y has two irreducible components A, D. We saw that $\phi_L(A)$ and $\phi_L(D)$ are rational normal curves and that $A \cong D \cong \mathbb{P}^1$. If $\phi_L(A) = \phi_L(D)$, then ϕ_L is the double covering we were looking for. Now assume that this is not the case. In this case (1) gives that ϕ_L is an isomorphism outside finitely many point. Hence $p_a(Y) \leq p_a(\phi_L(Y))$ and equality holds if and only if ϕ_L is an isomorphism. Since $\phi_L(A)$ and $\phi_L(D)$ are distinct rational normal curves, $\text{length}(\phi_L(A) \cap \phi_L(D)) \leq n+2$. Hence $p_a(\phi_L(Y)) \leq n+1 = (d-s)/2+1$. Thus $d-s = 2(g-s)-2$ and ϕ_L, ϕ_L is an isomorphism and $\text{length}(\phi_L(A)\cap\phi_L(D)) = n+2$. Since ϕ_L is an isomorphism, the linearly normal curve $C := \phi_L(Y) \subset \mathbb{P}^n$ has arithmetic genus $n-1$, degree $2n-2$ and each irreducible component of C spans $\mathbb{P}^n$. Riemann-Roch and duality for locally Cohen-Macaulay schemes gives $h^0(C, \mathcal{H}om(\mathcal{O}_C(1), \omega_C(1))) > 0$. Since C is a union of two rational normal curves intersecting in a length $n+2$ subscheme, we see that any non-zero section of $\mathcal{H}om(\mathcal{O}_C(1), \omega_C(1))$ is an isomorphism. Hence in this subcase Y is Gorenstein and $L \cong \mathcal{O}_Y$.

(vi) The degree 2 flat morphism $\psi : Y \to \mathbb{P}^1$ shows that Y has only planar singularities. Since X has either an ordinary node or an ordinary cusp at each point of S' and Y has only planar singularities, X has only planar singularities. □

Remark 2. Let Y be a reduced, connected and Gorenstein projective curve and $L \in \text{Pic}(Y)$. Set $y := \deg(L)$ and $m := h^0(Y, L) - 1$. Let $W_y^m(Y) \subseteq \text{Pic}^d(Y)$ denote the set of all degree y line bundles M on Y such that $h^0(Y, M) \geq m+1$. The algebraic set $W_y^m(Y)$ has a scheme-structure obtained in the following way. If Y is reducible, then $\text{Pic}^d(Y)$ has infinitely many connected components, labelled by the multidegrees with total degree d. It is sufficient to give a scheme-structure to the intersection of $W_y^m(Y)$ with any of these connected component. We may twist all line bundles in this component by a sufficiently ample line bundle on Y so that $h^1 = 0$ for all the twisted line bundles in this component. Then we may use the classical scheme-structure constructed in [1], pp. 176–177. Assume L spanned. Then the dimension of the cokernel of the multiplication map $\mu_L : H^0(Y, L) \otimes H^0(Y, \omega_Y \otimes L^*) \to H^0(Y, \omega_Y)$ is the dimension of the tangent space of $W^m_y(Y)$ at L ([1], [7], Example 3.3.9). Hence if μ_L has rank $\geq \rho$, then $W^m_y(Y)$ has dimension at most $p_a(Y) - \rho$ at L.

Proof of Theorem 1. Fix any $F \in \Gamma$ and let G_F be the subsheaf of F spanned by $H^0(X, F)$. Since We may find an algebraic partition of Γ into sheaves with the same singular support and such that the associated spanned subsheaves have constant degree. Since $\deg(G_F) \leq \deg(F)$ with strict inequality if and only if F is spanned and $X \backslash A$ is connected for every $A \subset S$, we reduce to the case $\mathrm{Sing}(F) = S$ and $G_F = F$ for all $F \in \Gamma$. Proposition 1 gives $h^0(X, F) \leq d - 2n - s$. Let $u : Y \to X$ be the partial normalization of X in which we only normalize the points of S. By assumption Y is connected and $p_a(Y) = g - s$. We may also assume that $m := h^0(X, F) - 1$ is the same for all $F \in \Gamma$. For any $F \in \Gamma$ set $L_F := u^*(F)/\mathrm{Tors}(u^*(F)$. Each L_F is a spanned line bundle on Y with degree $d - s$ and $h^0(Y, L_F) = h^0(X, F) = m + 1$ (Lemma 1). Since $u_*(L_F) \cong F$ (Lemma 1), Γ is an effective parameter space for the algebraic family $\{L_F\}_{F \in \Gamma}$ of line bundles on Y. Using Remark 2 we may copy the non-hyperelliptic part of a theorem of H. Martens given in [1], p. 193. $\square$

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