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ON THE DOMAIN OF THE CANONICAL MAP OF A STABLE POINTED CURVE

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Abstract: Let $Y = (X; P_1, \dots, P_n) \in \overline{\mathcal{M}}_{g,n}$ be a stable n-pointed curve. Here we describe the open subset $U_Y \subseteq X$ on which the pointed canonical line bundle $\omega_Y := \omega_X(P_1 + \cdots + P_n)$ is spanned.

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1. Introduction

Fix any pointed nodal curve $Y = (X; P_1, \dots, P_n) \in \overline{\mathcal{M}}_{g,n}$ with $g + n \geq 2$ and $n \geq 3$ if $g = 0$. X is a nodal and connected projective curve, $p_a(X) = g$, $P_1, \dots, P_n$ are distinct points of X_{reg} and each smooth and rational irreducible component of X contains at least 3 among the points in $\mathrm{Sing}(X) \cup \{P_1, \dots, P_n\}$. The canonical line bundle ω_Y of the pointed curve Y is the line bundle $\omega_X(P_1 + \cdots + P_n)$. The stability of the pointed curve Y is equivalent to the ampleness of ω_Y. There is a non-empty maximal open subset U_Y of X such that the complete linear system $|\omega_X(P_1 + \cdots + P_n)|$ induces a morphism $\phi_Y : U_Y \to \mathbb{P}^x$, where $x = g - 1$ if $n = 0$ and $x = g + n - 2$ if $n > 0$. In many cases (even with large g and $n = 0$ or with any g and large n) U_Y is not Zariski dense in X. The aim of this paper is to find U_Y and give some informations on the closure Y_ω of $\phi_Y(U_Y)$ in $\mathbb{P}^x$. As far as we know this was not known even in the classical case $n = 0$. The curve Y_ω may be called the *canonical image* of the pointed curve Y. The quasi-projective algebraic set $\phi_Y(U_Y)$ is called the

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effective canonical image of Y. A. Artamkin gave a necessary and sufficient condition for the equality $U_Y = X$ and a necessary and sufficient condition of the very ampleness (up to the "honestly hyperelliptic pointed curves" and two classified exceptional cases) of ω_Y ([1], Theorem 1.2; see [2], Theorems 3.3 and 3.6, for the case $n = 0$). Fix any pointed nodal curve $Y = (X; P_1, \dots, P_n)$ (even not connected). Let $\mathcal{B}(X)$ (resp. $\gamma(X)$ or $\gamma(Y)$, resp $\gamma'(Y)$) denote the set of all irreducible components (resp. connected components, resp. connected components containing no pointed point) of X. As in [1] we associate to Y the following modular graph $\|Y\|$. The set V_Y of the vertices of $\|Y\|$ is the set $\mathcal{B}(X)$. For any $v \in V_Y$ let T_v denote the corresponding irreducible component of X, while for every $T \in \mathcal{B}(X)$ let $[T]$ denote the corresponding vertex of $\|Y\|$. The set E_Y of edges of $\|Y\|$ is a disjoint union of two sets $\mathcal{S}_Y$ and $\mathcal{P}_Y$ where $\mathcal{S}_Y = \mathrm{Sing}(X)$ and $\mathcal{P}_Y = \{e_{P_1}, \dots, e_{P_n}\}$. If $P \in \mathrm{Sing}(X)$ lies in two irreducible components, say T, T' with $T \neq T'$, then the edge e_P has $[T]$ and $[T']$ as its vertices. If $P \in \mathrm{Sing}(X)$ lies in a unique irreducible component T of X, then e_P is a loop with $[T]$ as its vertex. The edges $e_{P_1}, \dots, e_{P_n}$ are called parabolic edges: they are not loops, but they only have one vertex. The parabolic edge e_{P_i} has as vertex the irreducible component of X containing P_i. A finite graph with possibly parabolic edges is called a marked graph. Set $U_Y := \{P \in X : \omega_Y$ is spanned at $P\}$. As in [1] a marked graph is said to be *compact* if it has no parabolic edge. A connected component of a marked graph is said to be *compact* if it contains no parabolic edge. For any marked graph Γ let $r(\Gamma)$ denote the number of the compact connected components of Γ. The *thickness* $t(\Gamma)$ of Γ is the minimal number of edges such that, removing at least one $A \subseteq E_Y$ with $\sharp(A) = t(\Gamma)$, the number of the compact connected components increases (see [1], p. 618). If $n \neq 1$ let Σ_Y be the set of all disconnecting nodes Q of X such that at least one of the connected components of $X \backslash \{Q\}$ contains no marked point. If $n = 1$ let Σ_Y be the union of $\{P_1\}$ and the set of all disconnecting nodes of X. Notice that $t(\|Y\|) = 1$ if and only if $\Sigma_Y \neq \emptyset$. Artamkin proved that $U_Y = X$ if and only if $\Sigma_Y = \emptyset$ ([1], part I of Theorem 1.2) and that $U_Y \subseteq X \backslash \Sigma_Y$ ([1], part 2) of Proposition 4.3 for the case $n = 0$, part 2) of Proposition 4.1 for the point P_1 if $n = 1$ and part 1) of Proposition 4.1 for the case $n \geq 2$. We may describe U_Y and the canonical linear system of Y in the following way.

Theorem 1. *Assume $g + n \geq 3$. Let B_Y be the union of all $T \in \mathcal{B}(X)$ such that $T \cong \mathbb{P}^1$ and $T \cap \overline{X \backslash T} \subseteq \Sigma_Y$. Let X_Y be the closure of $X \backslash B_Y$ in X. Set $Y' = (X_Y; Q_1, \dots, Q_m)$, where $\{Q_1, \dots, Q_m\} = \{P_1, \dots, P_n\} \cap X_Y$ and $Q_i \neq Q_j$ for all $i \neq j$. Let $v : D \to X$ be the partial normalization of X in which we only normalize the points of Σ_Y (case $n \neq 1$), or of $\Sigma_Y \backslash \{P_1\}$ (case*

$n = 1$). Let η be the union of the connected components of D isomorphic to $\mathbb{P}^1$ and (if $n \neq 1$) containing no point of $v^{-1}(\{P_1, \dots, P_n\})$. Set $C_Y := D \backslash \eta$ and $C'_Y := (C_Y; O_1, \dots, O_s)$, where $s = n = m$ and $\{O_1, \dots, O_s\} = \{P_1, \dots, P_n\}$ if $n \neq 1$, while $s = 0$ if $n = 1$. Then:

(a) if $n \neq 1$, then $m = n$ and $\{P_1, \dots, P_n\} = \{Q_1, \dots, Q_m\}$; if $n = 1$, then $m = 0$;

(b) $X \backslash U_Y = B_Y \cup \Sigma_Y$;

(c) $\omega_{C'_Y}$ is spanned and Y_ω is the image of C_Y by the complete linear system $|\omega_{C'_Y}|$.

Notice that B_Y is a union of some of the irreducible components of X and that each irreducible component of B_Y is isomorphic to $\mathbb{P}^1$. Each connected component of B_Y is a tree of $\mathbb{P}^1$'s. We call $Y' = (X_Y; Q_1, \dots, Q_m)$ the *abstract canonical pointed curve* of Y, $\omega_{Y'}$ the *spanned canonical line bundle* of Y, the morphism $h_{\omega_{Y'}} : C_Y \to \mathbb{P}^x$, $x = g - 1$ if $n = 0$, $x = g - 2 + n$ if $n > 0$, the *minimal canonical morphim* and its image Y_ω the *canonical image* of Y.

It is easy to give a recipe to construct all pointed curves with thicknees 1 in terms of pointed curves with thickness ≥ 2, some convention for the components of B_Y and the components of X mapped to a point of Y_ω, and some gluing data. For simplicity we first discuss the case $B_Y = \emptyset$ (see Construction 1) and then the general case (see Construction 2).

Corollary 1. *Fix $Y := (X; P_1, \dots, P_n) \in \overline{\mathcal{M}}_{g,n}$. U_Y is dense in X if and only if $B_Y = \emptyset$. Assume $B_Y = \emptyset$. Set $L_Y := \mathcal{I}_{\Sigma_Y} \otimes \omega_Y$. Then L_Y has pure rank 1 and it is the subsheaf of ω_Y spanned by $H^0(X, \omega_Y)$. If $n \neq 1$, then the natural canonical map $U_Y \to \mathbb{P}^x$ induces a morphism of a partial normalization of X and the minimal such partial normalization $u_Y : C_Y \to X$ is obtained normalizing only the points in Σ_Y. If $n = 1$, then U_Y induces a morphism $u_Y : C_Y \to X$, where u_Y is the partial normalization of X in which we normalize only the disconnecting nodes of X, i.e. the points of $\Sigma_{(X;\emptyset)}$. In both cases the morphism $C_Y \to \mathbb{P}^x$ with Y_ω as image is induced by the complete linear system $|M_Y|$, where M_Y is the line bundle $u_Y^*(L_Y)/\mathrm{Tors}(u_Y^*(L_Y))$ on C_Y.*

2. The Proofs

Abusing notation we will often write $(X; S)$ for a marked nodal curve, without specifying an ordering of the finite set $S \subset X_{reg}$. For any $L \in \mathrm{Pic}(X)$ we sometimes write $L(S)$ instead of $L(\sum_{P \in S} P)$. For any sheaf F with depth 1 and pure rank 1 on a nodal curve Z set $\mathrm{Sing}(F) := \{P \in Z : F$ is not locally free at $P\}$. Notice that $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(Z)$.

Remark 1. Let C be any nodal curve and $S \subset C_{reg}$ a finite subset. Fix a closed subcurve B of C and set $S' := B \cap S$. Fix any $S'' \subseteq S'$. The inclusion $j : B \hookrightarrow C$ is a finite morphism. Hence we may apply the adjunction formula for a finite morphism ([3], Example III.7.2) and get $\omega_B \cong Hom_{\mathcal{O}_C}(\mathcal{O}_B, \omega_C)$ and an inclusion $j_! : \omega_B \hookrightarrow \omega_C$ ([2], proof of Lemma 2.4). The inclusion $j_!$ induces an inclusion $\omega_B(S'') \hookrightarrow \omega_C(S)$. This is the inclusion implicitely used in part (c) of the statement of Theorem 1.

Remark 2. Let $Y = (X; P_1, \dots, P_n)$ be a semistable pointed curve and $Y_1 = (X_1; Q_1, \dots, Q_s)$ its stable reduction with $\pi : X \to X_1$ its associated contraction, with the convention that X_1 is a point if either $p_a(X) = 1$ and $n = 0$ or $X \cong \mathbb{P}^1$ and $n = 2$. Then $\pi_*(\omega_Y) \cong \omega_{Y_1}$, $\omega_Y \cong \pi^*(\omega_{Y_1})$, ω_Y is spanned if and only if ω_{Y_1} is spanned and $h^0(X, \omega_Y) = h^0(X_1, \omega_{Y_1})$. If $U_Y := \{P \in X : \omega_Y$ is spanned at $P\}$, then $U_Y = \pi^{-1}(\pi(U_Y))$, $\pi(U_Y) = \{P \in X_1 : \omega_{Y_1}$ is spanned at $P\}$, and the morphism $U_Y \to \mathbb{P}^x$, $x := p_a(X) - 1$, induced by $|\omega_Y|$ factors through π.

Remark 3. Let $Y = (X; P_1, \dots, P_n)$ be a connected pointed nodal curve. We stress that $P_j \in \Sigma_Y$ for some $j \in \{1, \dots, n\}$ if and only if $n = j = 1$, that $\Sigma_Y \subseteq \Sigma_{(X;\emptyset)}$ if $n \neq 1$ and that $\Sigma_Y = \Sigma_{(X;\emptyset)} \cup \{P_1\}$ if $n = 1$.

Lemma 1. *Let $Y = (X; P_1, \dots, P_n)$ be a connected pointed nodal curve. Fix $P \in \Sigma_Y$ with $P \notin \{P_1, \dots, P_n\}$ and let A_i, $i = 1, 2$, be the closures in X of the connected components of $X \backslash \{P\}$. Set $S_i := (\Sigma_Y \backslash \{P\}) \cap A_i$. Then $\Sigma_Y \backslash \{P\}$ is the disjoint union of the two sets $\Sigma_{(X_1;S_1)}$ and $\Sigma_{(X_2;S_2)}$, $S_1 \cup S_2 = \{P_1, \dots, P_n\}$, and either $S_1 = \emptyset$ or $S_2 = \emptyset$.*

Proof. The inclusion $\Sigma_{(X_1;S_1)} \cup \Sigma_{(X_2;S_2)} \subseteq \Sigma_Y \backslash \{P\}$ is obvious. Hence it is sufficient to prove the reverse containement. We may use induction on the integer $\sharp(\mathcal{B}(X))$, the result being obvious if $\sharp(\mathcal{B}(X)) \leq 2$. First assume $n = 0$. In this case the result is equivalent to the fact that $C_{(X,\emptyset)}$ has no disconnecting node, where $u_{(X,\emptyset)} : C_{(X,\emptyset)} \to X$ is the partial normalization of X in which we only normalize the disconnecting nodes. This fact is obvious and we also get that $C_{(X,\emptyset)}$ has $\sharp(\Sigma_{(X;\emptyset)}) + 1$ connected components. If $n = 1$ apply Remark 3 and then use the case $n = 0$ just stated. Now assume $n \geq 2$. The quasi-projective curve $X \backslash \{P\}$ has two connected components. The definition of thickness implies that either $S_1 = \emptyset$ or $S_2 = \emptyset$, say $S_1 = \emptyset$. Hence we may apply the case $n = 0$ to A_1 and see that it has no disconnecting node. Assume the existence of a disconnecting node Q of the pointed curve $(A_2; S_2)$.

Since $n \geq 2$, $Q \notin S_2$. The definition of thickeness shows that $A_2 \backslash \{Q\}$ has two connected components (call D_1 and D_2 their closure in A_2) and that one of these curves, say D_2, contains S_2. First assume $P \in D_1$. In this case $(A_1 \cup D_1)\backslash\{Q\}$ and $D_2\backslash\{Q\}$ are different connected components of $X\backslash\{Q\}$. Since $S_2 \cap (A_1 \cup D_2) = \emptyset$ and $S_2 = \{P_1, \dots, P_n\}$, we get $Q \in \Sigma_Y$. Now assume $P \in D_2$. In this case $X\backslash\{Q\}$ has $D_1\backslash\{Q\}$ and $(A_1 \cup D_2)\backslash\{Q\}$ as connected components. Since $S_2 \cap D_1 = \emptyset$, even in this case we get $Q \in \Sigma_Y$. □

Proof of Theorem 1. If $\Sigma_Y = \emptyset$, then ω_Y is spanned ([1], part I of Theorem 1.2). If $n = 1$, then we reduce to the case $(X_1; \emptyset)$, where X_1 is the stable reduction of X (notice that $\Sigma_X = \Sigma_{X_1}$ and use Remark 2). Hence we may assume $n \neq 1$, i.e. $\Sigma_Y \subseteq \Sigma_X \subseteq \mathrm{Sing}(X)$.

(i) Fix $P \in \Sigma_Y$ and let $u_P : C_P \to X$ be the partial normalization of X in which we normalize only the point P. The definition of thickness shows that $X\backslash\{P\}$ has two connected components and that one of them contains no marked point. Let D_1 and D_2 be the closure in X of these connected components with, say, $P_i \in D_2$ for all i. We have $\omega_Y|D_1 \cong \omega_{D_1}(P)$ and $\omega_Y|D_2 \cong \omega_{D_2}(P + P_1 + \cdots + P_n)$. Since $D_1 \cap D_2 = \{P\}$ (scheme-theoretic intersection) and ω_Y is a line bundle, we have the following Mayer-Vietoris exact sequence

$$0 \to \omega_Y \to \omega_Y|D_1 \oplus \omega_Y|D_2 \to \omega_Y|\{P\} \to 0\,. \tag{1}$$

Since P is a base point of $\omega_Y|D_1$, it is a base point of ω_{Y_1}. Since P is a base point both of ω_Y and $\omega_Y|D_1$, while $\omega_Y|D_1 \cong \omega_{D_1}(P)$ and $\omega_Y|D_2 \cong \omega_{D_2}(P+P_1+\cdots+P_n)$, (1) gives $x+1 = h^0(X, \omega_Y) = h^0(D_1, \omega_{D_1}) + h^0(D_2, \omega_{D_2}(P_1+\cdots+P_n))$. Set $L_Y := \mathcal{I}_{\Sigma_Y} \otimes \omega_Y$. Then L_Y is a subsheaf of ω_Y with pure rank 1. We just checked that the inclusion $L_Y \hookrightarrow \omega_Y$ induces an isomorphism $H^0(X, L_Y) \to H^0(X, \omega_Y)$. Let $v : D \to X$ be the partial normalization of X in which we normalize exactly the set Σ_Y. Set $M := v^*(L_Y)/\mathrm{Tors}(v^*(L_Y))$. Since $n \neq 1$, $\Sigma_Y \subseteq \Sigma_X \subseteq \mathrm{Sing}(X)$. Hence the definition of the sheaf L_Y gives $\Sigma_Y = \mathrm{Sing}(L_Y)$. Hence the classification of depth 1 modules with pure rank 1 on a nodal singularity ([4], pp. 164–166) gives $M \in \mathrm{Pic}(D)$ and $L_Y \cong v_*(M)$. The last assertion gives $h^0(X, L_Y) = h^0(D, M)$. Let Q_i, $1 \leq i \leq n$, be the only point of D such that $v(Q_i) = P_i$. Set $D' := (D; Q_1, \dots, Q_n)$. D' is a pointed nodal curve (not connected if $\Sigma_Y \neq \emptyset$). Fix $Q \in \Sigma_{D'}$ (if any). Since $n \neq 1$, $Q \in \mathrm{Sing}(D)$. By the definition of v we get $v(Q) \notin \Sigma_Y$. Hence $\{Q\} = v^{-1}(v(Q))$. Since $v(Q)$ does not disconnect Y, obviously Q does not disconnect D'. Hence $\Sigma_{D'} = \emptyset$. In particular no connected component of D contains a unique marked point Q_i. The subcurve B_Y is the image of the union η all connected components of D which are isomorphic to $\mathbb{P}^1$ and with no marked point, i.e. to all connected components of D' isomorphic to $(\mathbb{P}^1; \emptyset)$. The other connected components of D' are semistable and contain no disconnecting node. Hence $\omega_{A'}$ is spanned for every connected component A' of $D'\backslash\eta$. Notice that $M \cong \omega_{D'}$. Since $h^0(D, M) = h^0(X, L_Y)$, we get $x + 1 = h^0(D\backslash\eta, \omega_{D'\backslash\eta})$. Hence every point of $v(D\backslash\eta)\backslash\Sigma_Y$ is contained in U_Y. Since $v(D\backslash\eta)\backslash\Sigma_Y = X\backslash(\Sigma_Y \cup B_Y)$, we get $X\backslash U_Y \subseteq \Sigma_Y \cup B_Y$. Since the reverse inclusion is obvious, we get part (b). Part (a) follows from the proof of part (b) and the reduction of the case $n = 1$ to the case $n = 0$.

(ii) Here we check part (c). Recall that $h^0(X, M) = x+1$ and hat $M \cong \omega_{D'}$. Hence $h^0(\eta, M|\eta) = 0$ and $h^0(C_Y, \omega_{C'_Y}) = x + 1$. Let $f : C_Y \to \mathbb{P}^x$ denote the morphism induced by $|\omega_{C'_Y}|$. Every non-zero section α of ω_Y induces a non-zero section of L_Y and hence (since L_Y has no torsion) a non-zero section σ of M. Since $\sigma|\eta \cong 0$, α induces a non-zero section of $\omega_{C'_Y}$. Since $v : v^{-1}(U_Y) \to U_Y$ is an isomorphism, we see that $f \circ v^{-1} : U_Y \to \mathbb{P}^x$ in the morphism $\beta : U_Y \to \mathbb{P}^x$ associated to $|\omega_Y|$. Since $v^{-1}(U_Y)$ is dense in C_Y and $\omega_{CY'}$ is spanned, $f(C_Y)$ is the closure of $\beta(U_Y)$, i.e. $f(C_Y) = Y_\omega$. □

Proof of Corollary 1. Obviously, L_Y has pure rank 1, see the proof of Theorem 1. We proved that the inclusion map $L_Y \hookrightarrow \omega_Y$ induces an in isomorphism of global sections. Hence to prove the first assertion it is sufficient to prove that L_Y is spanned. By Theorem 1 ω_Y is spanned outside Σ_Y. Hence L_Y is spanned at each point of $X\backslash\Sigma_Y$. We saw that $L_Y = v_*(M)$ with M spanned. Fix $P \in \Sigma_Y$ and assume that L_Y is not spanned at P. Let N be the subsheaf of L_Y spanned by $H^0(X, L_Y)$. Since L_Y is spanned at each point of U_Y and U_Y is dense in X, N is a coherent sheaf on X with pure depth 1 Since $P \in \mathrm{Supp}(L_Y/N)$, and the connected component of ω_Y/L_Y supported by P is the length 1 skyscraper sheaf $\mathbb{K}_P$ supported by P, there is a length 2 zero-dimensional scheme $Z \subset X$ such that $Z_{red} = \{P\}$ and $N \subseteq \mathcal{I}_Z \otimes \omega_Y$. Since $h^0(X, N) = h^0(X, \omega_Y)$ it is sufficient to prove the inequality $h^0(X, \mathcal{I}_Z \otimes \omega_Y) < h^0(X, \omega_Y)$. Hence it is sufficient to prove $h^0(C_X, v^*(\mathcal{I}_Z \otimes \omega_Y)) < h^0(X, \omega_Y)$. Since M is spanned, to get the contradiction it is sufficient to note that $v^*(\mathcal{I}_Z \otimes \omega_Y))$ is a proper subsheaf of M. Notice that Z is a Cartier divisor of X. Hence $\mathcal{I}_Z \otimes \omega_Y$ is locally free. Hence $\mathrm{Supp}(M/v^*(\mathcal{I}_Z \otimes \omega_Y))$ contains one of the two pointsof $v^{-1}(P)$.

The other assertions were proved during the proof of Theorem 1. □

Now we give a recipe for the construction of connected pointed curves with thickness 1 in terms of connected pointed curves with thickness at least 2 and some gluing data. We first describe the case $B_Y = \emptyset$.

Construction 1. Any stable pointed curve Y with $B_Y = \emptyset$ may be obtained in the following way. Fix an integer $z \geq 2$. For each $i \in \{1, \ldots, z\}$ fix a connected pointed curve $A_i' = (A_i, P_{i,1}, \dots, P_{i,n_i})$ such that $A_i' \neq (\mathbb{P}^1, \emptyset)$, $n_i \neq 1$, and $t(\|A_i'\|) \geq 2$. We allow that case $p_a(A_i) = 1$ and $n_i = 0$. We set $C_Y' := \bigsqcup_{i=1}^{z} A_i'$ (disjoint union). We fix a function $\alpha : \{2, \dots, z\} \to \{1, \dots, z-1\}$ such that $\alpha(i) < i$ for all $i \in \{2, \dots, z\}$. For each $i \in \{2, \dots, x\}$ we fix $Q_{i1} \in (A_i)_{reg}$ and $Q_{i2} \in (A_{\alpha(i)})_{reg}$ with the restriction that no point Q_{ij}, $i \in \{2, \dots, z\}$, $j \in \{1, 2\}$, is a marked point and that all these points are different. The pointed curve Y is obtained from the pointed curve C_Y' gluing together each point Q_{i1} with the point Q_{i2}. Notice that $p_a(X) = \sum_{i=1}^{z} p_a(A_i)$.

Now we do the general case.

Construction 2. Take $\{A_i'\}_{1 \leq i \leq z}$ with $A_i' = (A_i, P_{i,1}, \dots, P_{i,n_i})$ as in Construction 1. Here we allow $(\mathbb{P}^1, \emptyset)$ for some i. We give the gluing data as in Construction 1. However, to have a stable pointed curve we need to impose that any component isomorphic to $(\mathbb{P}^1, \emptyset)$ is glued to at least 3 other components. We may first do the gluing concerning only pointed curves isomorphic to $(\mathbb{P}^1, \emptyset)$. In this way we obtain a precise description of the connected components of the subset B_Y of the pointed curve that we obtain in this way.

We are able to describe Y_ω in a few cases with $\|Y\|$ not 3-connected, but not yet in the general case. If $\|Y\|$ is m-connected with $m \geq 4$, then it is possible to describe what happens if Y_ω has an $(m-1)$-secant line.

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