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TWO PROJECTIONS AND ONE IDEMPOTENT

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Abstract: We state the unitary invariants of a pair of projections P and Q and its canonical representation. From this the unitary invariants and canonical form of idempotents are deduced. With this a number of well-known results on projections and idempotents can be derived easily. Finally quadratically normal operators are introduced and analyzed.

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1. Introduction

The spectral theorem is undoubtedly one of the cornerstones of Hilbert space theory. It gives the unitary invariants of a selfadjoint operator A on a Hilbert space $\mathcal{H}$ in terms of its multiplicity measure classes $\{\mu_n\}$ and provides unitarily equivalent model operators. The spectral theorem can also be used to derive the unitary invariants of a pair of projections P and Q in terms of a "generating" selfadjoint operator A, which also defines the operator models. Since the C^*-algebra $\mathcal{A}(E)$, generated by an idempotent E is also generated by the range projections P, Q of E respectively E^*, these invariants are unitary invariants of idempotents as well. With the aid of these models many results that appeared in the literature can be derived or generalized from a common point of view. This applies in particular to [5] which started our analysis. In contrast

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to previous papers, which rely quite a bit on algebraic manipulation, we rely essentially on the canonical representations of projections. In the last section these concepts will be extended to operators satisfying a quadratic identity. A number of these results are known [1, 2, 3, 5, 6, 7]. In Sections 2, 3 and 4, it is shown how a systematic use of the structure of two projections (see Theorem 2.1), considered as range projections of an operator A respectively A^* simplifies and shortens most proofs and makes those results more transparent. The results on quadratically normal operators are new. This paper is organized as follows: introduction, idempotents, operator algebras, further results, quadratically normal operators.

Throughout we will restrict ourselves to separable Hilbert spaces, even though the results can be upgraded to the non-separable case easily. The spectrum (point spectrum) of an operator A on $\mathcal{H}$ will be denoted by $\sigma(A)(\sigma_p(A))$. For operators $A, B, \ldots$ the C^*-algebra generated by $A, B, \ldots$ is $\mathcal{A}(A, B, \ldots)$. As usual M_k will denote the algebra of k by k complex matrices.

2. Idempotents

An operator A on a Hilbert space $\mathcal{H}$ will be called to be of type I_n, if $\mathcal{A}(A, 1)$ only has irreducible representations of dimensions less than n, i.e. $\hat{\mathcal{A}} =_n \hat{\mathcal{A}}$ [4, Chapter 3.6]. Selfadjoint operators are thus of type I_1. In this case there is a unitary map: $U : \mathcal{H} \to \sum^{\oplus} \mathcal{H}_i \otimes \mathbb{C}^i$ so that A becomes the sum of its “simple” multiplicity components

$$UAU^{-1} = \sum \oplus (A_i \otimes 1_i)\,. \tag{2.1}$$

Now let P and Q be two orthogonal projections. Then it is well known that $\mathcal{A} = \mathcal{A}(P, Q, 1)$ is of type I_2, that is, irreducible representations of $\mathcal{A}$ are either one or two dimensional. Moreover the generic two dimensional irreducible representation has the form

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} c^2 & sc \\ sc & s^2 \end{pmatrix}, \tag{2.2}$$

$$s = \sin\beta, c = \cos\beta, \quad 0 < \beta < \frac{\pi}{2}.$$

From this the following result can easily be deduced by standard C^*-algebra and von Neumann algebra techniques

Theorem 2.1. *Let P and Q be two orthogonal projections on the Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ has a unique decomposition $\mathcal{H} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{H}_2)$ so that:*

(i) $\mathcal{H}_1 = \{x \in \mathcal{H} | PQx = QPx\}$ *is a reducing subspace for* P *and* Q.

(ii) $P = P_1 + P_2$, $Q = Q_1 + Q_2$, *where* $P_1 = P \mid \mathcal{H}_1$, $P_2 = P \mid \mathcal{H}_1^{\perp}$ $Q_1 = Q \mid \mathcal{H}_1$ *and* $Q_2 = Q \mid \mathcal{H}_1^{\perp}$. $\mathcal{H}_2 = P_2 \mid \mathcal{H}_1^{\perp}$.

(iii) There exists a selfadjoint operator A *on* $\mathcal{H}_2$ *with* $\sigma(A) \subset \left[0, \frac{\pi}{2}\right]$, *and* $0, \frac{\pi}{2} \notin \sigma_p(A)$ *so that* P *and* Q *are unitarily equivalent to*

$$P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}, \tag{2.3}$$

$$c = \cos\ A, \quad s = \sin\ A\,.$$

(iv) The unitary invariants of P *and* Q *are those of* A *and the multiplicities*

$$\dim(\mathcal{H}_1 \ominus (P_1 + Q_1 - P_1Q_1)\mathcal{H}_1), \quad \dim(P_1 - P_1Q_1), \tag{2.4}$$

$$\dim(Q_1 - P_1Q_1), \quad \dim(P_1.Q_1).$$

In the remainder, Theorem 2.1 will be used systematically to obtain information about operators A for which P and Q are the range projections of A respectively A^*. As a rule, statements about the commutative parts P_1, Q_1 tend to be trivial.

Let E be an idempotent on the Hilbert space H and let P and Q be the range projections of E respectively E^*, i.e. $PEE^* = EE^*P$ and $QE^*E = E^*EQ$. Then $((1-P)E)(I-P)E)^* = 0$ shows $PE = E$ and by symmetry

$$PEQ = E\,. \tag{2.5}$$

As before write $\mathcal{H} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{H}_2)$ and $P = P_1 + P_2, Q = Q_1 + Q_2$ for the decomposition of $\mathcal{H}, P$ and Q into its commutative and genuinely type I_2 parts.

Theorem 2.2. *a) A pair of projections* P, Q *is the range projections of* E *respectively* E^* *iff* $P_1Q_1 = P_1 = Q_1$ *and if the operator* A *of Theorem 2.1 satisfies* $||A|| < \frac{\pi}{2}$.

b) $\mathcal{A}(P, Q, 1) = \mathcal{A}(E, 1)$ *and the unitary invariants of* E *are those of* P, Q, *i.e. dim* P_1Q_1, *dim* $\mathcal{H}_1-$ *dim*P_1Q_1 *and those of* A.

c) Any idempotent E *on* $\mathcal{H}$ *is unitarily equivalent to an operator of the form*

$$E = 0\ \oplus\ 1\ \oplus \begin{pmatrix} 1 & tgA \\ 0 & 0 \end{pmatrix} \quad \text{on } (\mathcal{H}_{10} \oplus \mathcal{H}_{11}) \oplus (\mathcal{H}_2 \oplus \mathcal{H}_2)\,, \tag{2.6}$$

where $\mathcal{H}_1 = \mathcal{H}_{10} \oplus \mathcal{H}_{11}$ *and where* A *satisfies* $A = A^*, ||A|| < \frac{\pi}{2}$ *with* $0 \notin \sigma_p(A)$.

Proof. a) Let P and Q be the range projections of E respectively E^* and let $R = 1-(P_1-P_1Q_1)-(Q_1-P_1Q_1)$. Then $E = E^2 = EQ(1-R)PE+EQRPE = ERE$ because $Q(1-R)P = Q_1(1-R)P_1 = 0$. Let $F = ER$, then it follows from $ERE = E$ that F is an idempotent with range projection P. Hence if $P_1 > P_1Q_1$ there exists a $0 \neq x \in (P_1 - P_1Q_1)\mathcal{H}$. Since x is in the range of F we have $Fx = x$ contradicting $Rx = 0$. For $Q_1 = P_1Q_1$ argue similarly. Any $x \in P_1Q_1\mathcal{H}$ is left invariant by E as well as E^*, by symmetry. Thus $P_1Q_1\mathcal{H}$ is a reducing subspace of E. So is the common nullspace of E and E^*, $\mathcal{H}_1 \ominus P_1Q_1\mathcal{H}$. Thus we can decompose $\mathcal{H}$ as in Theorem 2.1 into reducing subspaces of E. The corresponding decompositon of E is then

$$E = 0 \oplus 1 \oplus E_2$$

In order to find a representation of E_2 use an operator matrix representation of E_2 on $(\mathcal{H}_2 \oplus \mathcal{H}_2)$ together with (2.5) and an operator matrix representation for P_2 and Q_2 (2.2).

b) This shows (b) as well as (2.6). This expression, however, shows $\mathcal{A}(P, Q, 1) = \mathcal{A}(E, 1)$.

c) It follows from (2.6) that

$$||E||^2 = ||EE^*|| = ||1 + tg^2 A|| = \frac{1}{\cos^2 \|A\|} \tag{2.7}$$

by the spectral mapping theorem. This shows $\sigma(A) \subset [0, \beta], \beta < \frac{\pi}{2}$. □

Theorem 2.2 shows that the idempotent E is similar to the projection $P_1 + P_2$. By induction one can then show that a finite family of orthogonal idempotents is similar to a finite family of orthogonal projections.

3. Operator Algebras

In Theorem 2.1 we have described two projections P and Q via their commutative parts P_1 and Q_1 and via a selfadjoint operator A with $\sigma(A) \subset [0, \frac{\pi}{2}]$ with $0, \frac{\pi}{2} \notin \sigma_p(A)$. Straightforward functional calculus now shows that the non-commutative part of $\mathcal{A} = \mathcal{A}(P, Q, 1)$, the core part is the unitization of the algebra of continuous M_2-valued functions f on $\sigma(A)$, which vanish at 0. Thus $\hat{\mathcal{A}}_2$, the open part of $\hat{\mathcal{A}}$ corresponding to two dimensional irreducible representations, can be identified with $\sigma(A) \backslash \{0, \frac{\pi}{2}\}$. The topology on $\hat{\mathcal{A}}_2$ is that of $\sigma(A) \backslash \{0, 1\} \subset \mathbb{R}$, because the centre of this part of $\mathcal{A}$ is generated by $(P - Q)^2 = \sin^2(A)$. Attached to $\hat{\mathcal{A}}_2$ are possibly four one-dimensional representations. These correspond to 0 with $P_1 = Q_1 = 0$ or 1 and at $\frac{\pi}{2}, P_1 = 1, 0\ , Q_1 = 0, 1$. Their presence can be inferred from the sign of the multiplicities (2.4). The corresponding JB*-algebra generated by P, Q and 1 is thus the selfadjoint part of $\mathcal{A}(P, Q, 1)$.

For an idempotent E with associated range projections P and Q we have $P_1 = P_1\ Q_1 = Q_1$ so that either $\mathcal{A} = \mathcal{A}(E)$ is the unitization of the core part as above or this plus one one-dimensional representation corresponding to $P_1 = Q_1 = 1$ attached to it. In addition $\sigma(A)$ is bounded away from $\frac{\pi}{2}$. This also shows that up to C*-isomorphism $\mathcal{A}$ is independent on the multiplicities and the multiplicities in (2.4) may be chosen as either 0 or 1 when studying $\mathcal{A}^*(P, Q, 1)$.

4. Further Results

In this section we will derive some known results for idempotents and projections in a direct and simple manner from Theorems 2.1 and 2.2. Thus we will use the same notation as above and write $P = P_1 + P_2, Q = Q_1 + Q_2$ for the canonical decomposition of P and Q into their commutative and genuinely I_2 parts. Similarly we will write $E = E_1 + E_2$. In general the proofs for the commutative parts on $\mathcal{H}_1$ are rather trivial so that in most cases one may assume $\mathcal{H} = \mathcal{H}_2 \oplus \mathcal{H}_2$ and (2.4).

Lemma 4.1. *$P - Q$ is invertible iff $P_1Q_1 = 0$ and $0 \notin \sigma(A)$. In this case $\mathcal{H}$ is the direct sum of $P\mathcal{H}$ and $Q\mathcal{H}$.*

Proof. The condition $P_1Q_1 = 0$ is obviously necessary and sufficient for the commutative part. Thus we may assume $\mathcal{H}_1 = 0$. Now $P - Q = s.\begin{pmatrix} s & -c \\ -c & -s \end{pmatrix}$ is invertible iff $s = \sin A$ is invertible, because $\begin{pmatrix} s & -c \\ -c & -s \end{pmatrix}$ is invertible. This, however, is equivalent with the invertibility of A. Clearly $\mathcal{H} = P\mathcal{H} \oplus (1 - P)Q(1 - P)\mathcal{H}$ iff s respectively A are surjective. □

Somewhat more general is the question when $P - Q$ is a Fredholm operator. Again, since the indices are additive with respect to the decompositions into the commutative and type I_2 parts and (2.1), it suffices to study all summands separately. The commutative part is Fredholm of index 0 iff dim $P_1Q_1 < \infty$. As regards the type I_2 part, $P_2 - Q_2 = \begin{pmatrix} s^2 & -cs \\ -cs & -s^2 \end{pmatrix} = s \cdot \begin{pmatrix} s & -c \\ -c & -s \end{pmatrix}$ is Fredholm of index 0 iff $s = \sin\ A$ is Fredholm, because $D = \begin{pmatrix} s & -c \\ -c & -s \end{pmatrix}$ is invertible,$D^2 = 1$. $s = \sin A$ on $\mathcal{H}_2$ is a Fredholm operator iff A is. This in turn holds iff the essential spectrum $\sigma_{ess}(A)$ is bounded away from 0. With this it is easy to derive the results in [6] via a compact perturbation of A.

Let P and Q be projections with $||P-Q||<1$. Then $P_1=P_1Q_1=Q_1$ and there is an idempotent E for which P and Q are the range projections. In this case

$$||E||^2=(\cos\|A\|)^{-2}=(1-\|s\|^2)^{-1}=(1-\|(1-P_2)Q_2\|^2)^{-1} \tag{4.1}$$

because $\|(1-P_2)Q_2\|^2=\|\begin{pmatrix}0&0\\cs&s^2\end{pmatrix}\|^2=\|s\|^2$ by (2.7). This is the main result of [8].

Finally let us analyze the situation, when $P-Q\in\mathcal{K}(\mathcal{H})$ the algebra of compact operators on $\mathcal{H}$. For the type I_1 part this requires $P_1-P_1Q_1$ and $Q_1-P_1Q_1$ to be projections of finite rank. Arguing as above we see $P_2-Q_2\in\mathcal{K}(\mathcal{H})$ iff $s\in\mathcal{K}(\mathcal{H}_2)$ or $A\in\mathcal{K}(\mathcal{H}_2)$. In this case the discrete spectral decompositon of A leads to the representation (2.2) $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_1^\perp$ with $\mathcal{H}_1^\perp=\sum\oplus\mathbb{C}^2$ and

$$\begin{aligned}&P_2=\sum\oplus\begin{pmatrix}1&0\\0&0\end{pmatrix},Q_2=\sum\oplus\begin{pmatrix}c_i^2&c_is_i\\c_is_i&s_i^2\end{pmatrix},\\&0<\lambda_i<\frac{\pi}{2},\quad\lambda_i\to0,\quad s_i=\sin\lambda_i,c_i=\cos\lambda_i,\lambda_i\in\sigma(A).\end{aligned} \tag{4.2}$$

Then the trace of $P-Q$ is

$$tr(P-Q)=tr(P_1-Q_1)+\sum tr\begin{pmatrix}s_i^2&-c_is_i\\-c_is_i&-s_i^2\end{pmatrix}=tr(P_1-Q_1)$$

is an integer. Moreover we see with $s=s_i$

$$(P_i-Q_i)^{2n}=\begin{pmatrix}s_i^2&-c_is_i\\-c_is_i&-s_i^2\end{pmatrix}^{2n}=s_i^{2n}\begin{pmatrix}s_i&-c_i\\-c_i&-s_i\end{pmatrix}^{2n}=s_i^{2n}\cdot1\,.$$

Thus

$$tr(P-Q)^{2n+1}=tr(P-Q)^{2m+1}=tr(P_1-Q_1),\quad m\geq n \tag{4.3}$$

if $P-Q$ respectively A belongs to the Schatten class $\mathcal{C}_{2n+1}$, i.e. $\sum|\lambda_i|^{2n+1}<\infty$. This result has recently been shown by Avron, Seiler and Simon [1]. We add that in this case

$$tr(P-Q)^{2n}=\dim(P_1-P_1Q_1)+\dim(Q_1-P_1Q_1)+2\sum\sin^{2n}\lambda_i\,. \tag{4.4}$$

Let P and Q be idempotents with $P_1=P_1Q_1=Q_1$, then there exists a unitary operator W with

$$WPW^{-1}=Q\text{ and }WQW^{-1}=P\,. \tag{4.5}$$

With $W/\mathcal{H}_1=1$ it suffices to solve the problem for the I_2 part. There by the functional calculus establishd above, it suffices to solve the problem on $\mathbb{C}^2$. Here W is just the reflection on the line halving the angle between $P\mathcal{H}$ and $Q\mathcal{H}$. Explicitly W is given by $W=\begin{pmatrix}-c&-s\\-s&c\end{pmatrix}$ with $c=\cos\ A,s=\sin\ A$

as above. These results are some of the key results in [1]. It follows from (4.5) that $WEW^{-1} = E^*$ for the idempotent E associated to P and Q.

Let P and Q be projections with $||P - Q|| < 1$. Then $P_1 = P_1Q_1 = Q_1$ and there is an idempotent E with associated range projections P and Q.

In [1] Avron, Seiler and Simon call a pair of projections Fredholm if $QP : P\mathcal{H} \to Q\mathcal{H}$ is a Fredholm operator. This is clearly equivalent to (P_1, Q_1) and (P_2, Q_2) being Fredholm. (P_1, Q_1) is Fredholm iff $\dim ker(Q_1, P_1) = \alpha = \dim(P_1 - Q_1P_1)$ and $co\dim Q_1P_1 = \dim Q_1 - P_1Q_1 = \beta$ are finite dimensional, the index then being the difference of these numbers. For the noncommutative part use again the matrix representation (2.3). Then

$$QP = \begin{pmatrix} c^2 & 0 \\ cs & 0 \end{pmatrix}, (P - Q - 1) = \begin{pmatrix} -c^2 & -cs \\ -cs & -(1 + s^2) \end{pmatrix} \text{ and}$$

$$(P - Q + 1) = \begin{pmatrix} 1 + s^2 & -cs \\ -cs & c^2 \end{pmatrix}.$$

Thus the range of Q_2P_2 is closed only if this holds for c, or $0 \notin \sigma(c)$ and $||s|| < 1$. Thus $||(P_2 - Q_2)|| < 1$ so that 1 and -1 are isolated in $\sigma(P - Q)$. In fact they arise only from the commutative part. Moreover index $(P_2, Q_2) = 0$.

Thus the nontrivial index of (P, Q) arises only from its commutative part, while the range conditions fix the spectral condition for its noncommutative part. If (P, Q) is Fredholm $(P_1 - Q_1)$ is of finite rank, while

$$P_2 - Q_2 = s \begin{pmatrix} s & -c \\ -c & -s \end{pmatrix}$$

is a selfadjoint operator with $||P_2 - Q_2|| = ||s|| < 1$ (see [1], Proposition 3.2). Now index $(P, Q) = index(P_1, Q_1) = -index(Q_1, P_1) = -index(Q, P)$ proves Theorem 3.4 of [1].

5. Quadratically Normal Operators

Let A be an operator on $\mathcal{H}$ satisfying

$$A^2 + aA + b = 0 \tag{5.1}$$

and let $p(x) = x^2 + ax + b$. Then we have two cases to consider. In the first case p has two distinct zeros. By scaling, this can be reduced to $p = x^2 - x$. On the operator level this leads to idempotents which have been analyzed above. In the second case, where p has a double zero, scaling leads to $A^2 = 0$. Thus let A be an operator $\mathcal{H}$ with $A^2 = 0$. Let P and Q be the range projections of A respectively A^*. Then $PAQ = A$ and $P \cdot Q = 0$. Now consider $B = P + A$.

Then $B^2 = (P + A)^2 = P + PA + AQ + A^2 = B$. So B is an idempotent.

Thus operators satisfying a quadratic identity are operators of type I_2. Now let A be an operator which commutes with $A^{*2} + \overline{a}A^*$. Then $A^2 + aA = N$ is a normal operator. Let π be an irreducible representation of $\mathcal{A}(A)$. Then $\pi(N)$ is in the centre of $\pi(\mathcal{A}(A))$. Thus $\pi(A)^2 + a\pi(A) = \lambda_\pi$ and $\pi(A)$ satisfies a quadratic identity. Thus $\dim \pi \leq 2$. This shows

Theorem 5.1. *An operator A, which commutes with $A^{*2} + \overline{a}A^*$ is a type I_2 operator.*

We will call such operators quadratically normal, because then $A^2 + aA = N$ is normal. A simple linear transformation reduces the analysis to the case where $A^2 = N$ is normal. For simplicity, however, we consider the equation

$$A^2 = N^2 \tag{5.2}$$

because standard functional calculus in C*-algebras permits square roots. In this case we even choose A so that $A_1 = N|\mathcal{H}_1$. It follows from (5.2) that N belongs to the centre $\mathcal{Z}$ of $\mathcal{A}(A)$.

Now let

$$Z = (A + N)(A + N)^* + (A - N)^*(A - N)\,. \tag{5.3}$$

Lemma 5.2. *$Z \in \mathcal{Z}, Z$ is invariant under $N \to -N$.*

Proof. We may assume that A acts irreducibly on $\mathbb{C}^2$. Thus $N^2 = \lambda^2 = A^2$ for some λ. Let P and Q be the range projections of $A + \lambda$ and $(A + \lambda)^*$ respectively. Then

$$P(A + \lambda)Q = (A + \lambda)$$

with $c = \cos\ S, s = \sin\ S$ and $S = S^*$ with $\sigma(S) \subset (0, \frac{\pi}{2})$ and the representation $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$ leads in the case of $\lambda \neq 0$ to the form

$$A = \begin{pmatrix} \lambda & 2\lambda\ tg \\ 0 & -\lambda \end{pmatrix} \text{ with } tg = \frac{s}{c}, s = \sin\ S, c = \cos\ S, \text{ as before }. \tag{5.4}$$

This then shows

$$Z = 4NN^*(1 + tg^2) \cdot 1\,. \tag{5.5}$$

In order to see whether N and Z generate the centre $\mathcal{Z}$ in $\mathcal{A}$, consider a representation of $\mathcal{A}$ for which $Z = \mu^2$ and $N = \lambda$. Then (5.3) holds and by (5.5) S is a scalar. Thus $\hat{\mathcal{A}}_2 = \hat{\mathcal{Z}}$. This also shows that the invariants of quadratically normal operator A are the invariants of N and Z. These are not quite independent however. The C^*-algebra generated by such an operator can be identified with matrix valued functions on the spectrum of $\mathcal{Z}$ the centre of $\mathcal{A}$. The values will be two by two matrices unless $Z = NN^*$. It is clear now, how to derive the unitary invariants for A from N and Z. The following examples show that the situation for non type I_2 operators becomes rather hopeless. $\square$

Example 1. There exists an antiliminal C^*- algebra generated by two orthogonal projections P_1, P_2 and a projection Q. To see this let

$$U = P_1 + P_2 e^{\frac{2\pi i}{3}} + (1 - P_1 - P_2)e^{\frac{4\pi i}{3}} \text{ and } V = 1 - 2Q\,.$$

Then $U^3 = 1 = V^2$ and U and V define a representation of $\mathbb{Z}_3 * \mathbb{Z}_2$, the free product of $\mathbb{Z}_3$ with $\mathbb{Z}_2$. This group is not of type I, because it is not a finite dimensional extension of an Abelian group.

Example 2. On $\mathcal{H} = \mathcal{H}_1 \otimes \mathbb{C}^3$ let $A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$ with a, b positive invertible. Then $A^3 = 0$ and $\mathcal{A}(A) = \mathcal{A}(a,b) \otimes M_3$. Thus $\mathcal{A}(A)$ is antiliminal if $\mathcal{A}(a,b)$ is. This is easy to achieve, e.g. using Example 1.

Example 3. For the orthogonal projection $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and the idempotent $Q = \begin{pmatrix} 0 & a & ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ one has $QP = 0$. a, b may be chosen such that $\mathcal{A}(P,Q)$ is antiliminal.

Example 4. On $\mathcal{H} = \mathcal{H}_2 \oplus \mathcal{H}_2$ the operator matrix $E = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix}$ with $A \geq 0$ selfadjoint and domain of definition $D = \{(x,y)^t | x \in \mathcal{H}_2, y \in D_A\}$ defines a possibly unbounded closed idempotent. Such operators have apparently not been studied before.

Example 5. On $l^2(\mathbb{Z})$ consider the bilaterial shift U to the right, $Ue_n = e_{n+1}, n \in \mathbb{Z}$. Let P be the projection onto the subspace generated by the basis elements $e_n, n \geq 0$. Then $UPU^* = Q$ projects onto the subspace generated by $e_n, n \geq 1$. Thus $P = P_1$ and $Q = Q_1 = P_1 Q_1$ and index $(P,Q) = 1$. By taking direct sums and replacing U by U^* it is thus possible to create projections $(P, UPU^* = Q)$ of arbitrary index. Indications that this is typical come from the following observations.

A Fredholm pair (P,Q) can always be written as a restriction of a maximal pair, i.e. a pair of which either $\alpha(P,Q)$ or $\beta(P,Q) = 0$. The noncommutative part can be deformed continuously until the pair is commutative. Commutative

Fredholm pairs, however, can easily seen to be sums of the above form. Such ideas can likewise be applied within the concept of unitary deformations. The question of fractional indices would then lead to problems of constructing roots of the bilateral shift.

Remark. In a sense Example 5 is typical in the domain of unitary deformations, because if $(P, UPU^* = Q)$ is such a Fredholm pair our results above directly lead to a unitary V, which satisfies $V|\mathcal{H}_1 = 1$ and $V_2 = V|\mathcal{H}_2$ satisfying $V_2 Q_2 V_2^* = P_2$. Then $W = VU$ leads to the projection $\tilde{Q} = WPW^*$ which commutes with P so that $index(P, Q) = index(P, \tilde{Q})$. If the unitary U has a gap in the spectrum it may be embedded into a continuous unitary group (U_t) so that a continuity argument shows $index(P, UPU) = 0$.

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