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FIXED POINT RESULTS FOR THE BANACH SPACE OF HERMITIAN ELEMENTS OF A BANACH ALGEBRA

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Abstract: The paper studies the convergence of Krasnoselskij iterative process to fixed points of norm-decreasing isomorphisms on the space of Hermitian elements of a uniformly convex Banach algebra.

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1. Introduction

Nonexpansive mappings, T, have been widely studied in relation to the existence of fixed points in normed spaces. Among several authors who have contributed to this study are Browder [4], Karlovitz [6], Kirk [7], Mijajlovic [9] and others. Closely related to nonexpansive mappings are norm-decreasing operators in Banach algebras. This paper establishes convergence results for a norm-decreasing isomorphism T of a Banach algebra A. The motivation for this work comes from the interesting paper of Oshobi [10].

A normed algebra is unital (or norm-unital) if it contains an identity element which has norm one. We denote such identity by 1. Let A be a unital normed algebra. The set $\{w \in A^* : w(1) = \|w\| = 1\}$ is denoted by A^*_W. For any

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element $a \in A$ the numerical range $W(a)$ of a is the set $\{w(a) : w \in A_W^*\}$ and the number $\|a\|_W = \sup\{|\lambda| : \lambda \in W(a)\}$ is called the numerical radius of a. We recall that $W(a)$ is a nonempty, compact, convex subset of the complex plane. Also if A is a unital Banach algebra, then $\|a\|_W \leq \|a\|$.

Definition 1.1. An element h of a unital Banach algebra A is called Hermitian if its numerical range is included in the real axis. We denote by $H(a)$ the set of Hermitian elements in A.

The following result is presented in Bonsall [3].

Lemma 1.2. *Let A be a norm-unital Banach algebra. The following conditions are equivalent for an element $h \in A$:*

(i) $h \in H(a)$,

(ii) $\left\|e^{i\alpha h}\right\| = 1 \quad \forall \quad \alpha \in \Re$,

(iii) $\lim\limits_{\alpha \to 0} \frac{1}{\alpha}\{\|1 - i\alpha h\| - 1\} = 0$.

Furthermore, if $\rho(h)$ represents the spectral radius of h, then

(iv) $\rho(h) = \|h\|_W = \|h\|$.

Remark 1.3. An operator $T : A \to A$ is said to be norm-decreasing if $\|Tx\| \leq \|x\|$. However, it is easy to see that every norm-decreasing operator is nonexpansive.

2. Main Results

Our main results are presented below.

Theorem 2.1. *Let A be a unital Banach algebra. Then $H(A)$ is a closed, bounded and convex Banach space.*

Proof. We first observe that $H(A)$ is a real linear space which is closed in the weak topology and therefore closed in the norm topology of A. Moreover, the boundedness of $H(A)$ follows from that of $W(a)$ and the condition (iv) above.

Indeed, $\lambda \in W(a)$ implies $|\lambda| \leq M < \infty$. But $\|h\| = \|a\|_W = \sup\{|\lambda| : \lambda \in W(a)\}$, $h \in H(A)$. Therefore, $\|h\| \leq M < \infty$.

We now show that if $x,\ y\ \in H(A)$, then $s = \lambda x + (1-\lambda)y \in H(A)$ whenever $\lambda \in [0, 1]$.

Now, since $\alpha\lambda,\ \ \alpha(1-\lambda) \in \Re$, then by Lemma 1.2, we have

$$\left\|e^{i\alpha s}\right\| = \left\|e^{i\alpha\lambda x}.e^{i\alpha(1-\lambda)y}\right\| \leq \left\|e^{i\alpha\lambda x}\right\| \left\|e^{i\alpha(1-\lambda)y}\right\| = 1.$$

On the other hand, from $1 = \|1\| = \left\|e^{i\alpha s}.e^{-i\alpha s}\right\| \leq \left\|e^{i\alpha s}\right\| \left\|e^{-i\alpha s}\right\|$, we obtain,

$$\frac{1}{\|e^{i\alpha s}\|} \leq \left\|e^{-i\alpha s}\right\| = \left\|e^{-i\alpha\lambda x}.e^{-i\alpha(1-\lambda)y}\right\| \leq \left\|e^{-i\alpha\lambda x}\right\| \left\|e^{-i\alpha(1-\lambda)y}\right\| = 1.$$

These yield $\left\|e^{i\alpha s}\right\| \leq 1 \leq \left\|e^{i\alpha s}\right\|$, which means $\left\|e^{i\alpha s}\right\| = 1$. Thus, $s \in H(A)$.

Finally, $H(A)$ is a Banach space follows from Proposition 2.3 of [10].

The following forms a part of Oshobi's results in [10].

Theorem 2.2. (see Oshobi [10]) *Let T be a norm decreasing isomorphism of a complex unital Banach algebra A onto another B, then $TH(A) \subseteq H(B)$.*

The following is a fixed point result in $H(A)$.

Theorem 2.3. *Let A be a complex unital Banach algebra that is uniformly convex. Let $T : A \to A$ be a norm-decreasing isomorphism and $T_{H(A)}$ the restiction of T to H(A), then the sequence $\{x_n\}_{n=0}^{\infty} \subseteq H(A)$ defined by $x_{n+1} = \lambda x_n + (1-\lambda)T_{H(A)}x_n$, converges to the fixed point of $T_{H(A)}$.*

Proof. For the sake of simplicity, we write T_H instead of $T_{H(A)}$. We note that T_H is a selfmap of $H(A)$ by Theorem 2.2.

The Brouwer's Fixed Point Theorem guarantees that the set of fixed points F_{T_H} of T_H is nonempty. Now let $p \in F_{T_H}$,

$$\begin{aligned}\|x_{n+1} - p\| &= \|\lambda(x_n - p) + (1-\lambda)(T_H x_n - p)\| \\ &\leq \lambda \|x_n - p\| + (1-\lambda) \|x_n - p\| = \|x_n - p\|.\end{aligned}$$

That is, the sequence $\{\|x_{n+1} - p\|\} \subset \mathbf{R}$ is monotonic decreasing, and hence converges to a real number a.

Suppose $a > 0$. Since $\|x_n - T_H x_n\| \leq \|x_n - p\| + \|p - T_H x_n\| \leq 2\|x_n - p\|$, $\{\|x_{n+1} - p\|\}$ is a bounded sequence. It was shown (see Rhoades [11]), using the lemma of Groetsch, that $\lim \|x_{n+1} - T_H x_n\| = 0$. Since $T(H(A))$ is compact there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $\lim_i Tx_{n_i} = u \in H(A)$.

Therefore,

$$\|x_{n_i} - u\| \leq \|x_{n_i} - T_H x_{n_i}\| + \|T_H x_{n_i} - u\| \to 0 \text{ as } i \to \infty.$$

But T_H is nonexpansive, so we have $\|T_H x_{n_i} - T_H u\| \leq \|x_{n_i} - u\| \to 0$ as $i \to \infty$.

That is, $T_H u = u$. If we now replace p by u in the early part of the proof, we see that $\{\|x_n - u\|\}$ is monotone decreasing in n. Since $\lim_i x_{n_i} = u$, it means $\lim x_n = u$, and $\{x_n\}$ converges to a fixed point of T. □

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