PRIME RADICAL OF ORE EXTENSIONS OVER $\delta$-RINGS

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Abstract: Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$. We recall the definition of a $\sigma(*)$-ring, and find a relation between the prime radical of a $\sigma(*)$-ring $R$ and that of $R[x;\sigma]$. Let now $\delta$ be a $\sigma$-derivation of $R$. We say that a ring $R$ is a $\delta$-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, $a \in R$; where $P(R)$ is the prime radical of $R$. We then find a relation between the prime radical of a $\delta$-ring $R$ and that of $R[x;\sigma;\delta]$. We generalize the result for a Noetherian $Q$-algebra ($Q$ is the field of rational numbers).

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1. Introduction

A ring $R$ always means an associative ring. The field of rational numbers is denoted by $Q$, and the ring of integers is denoted by $Z$ unless other wise stated. The set of all prime ideals of $R$ is denoted by $\text{Spec}(R)$ and the sets of all minimal prime ideals of $R$ is denoted by $\text{Min.}(\text{Spec}(R))$. The prime radical and the set of all nilpotent elements of $R$ are denoted by $P(R)$ and $N(R)$ respectively.

Let $R$ be a ring. Let $\sigma$ be an automorphism and $\delta$ be a $\sigma$-derivation of $R$. Recall that $R[x;\sigma;\delta]$ is the usual polynomial ring with coefficients in $R$ and we consider any $f(x) \in R[x;\sigma;\delta]$ to be of the form $f(x) = \sum x^i a_i$, $0 \leq i \leq n$. Multiplication in $R[x;\sigma;\delta]$ is subject to the relation $ax = x\sigma(a) + \delta(a)$ for $a \in R$.

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Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors, see [1], [2], [3], [6], [10], [11]. In [1] associated prime ideals of skew polynomial rings have been discussed. In [3] it is shown that if R is embeddable in a right Artinian ring and if characteristic of R is zero, then the differential operator ring $R[x; \delta]$ embeds in a right Artinian ring, where $\delta$ is a derivation of R. It is also shown in [3] that if R is a commutative Noetherian ring and $\sigma$ is an automorphism of R, then the skew-polynomial ring $R[x; \sigma]$ embeds in an Artinian ring. In [2] it is proved that if R is a ring which is an order in an Artinian ring, then $R[x; \sigma; \delta]$ is also an order in an Artinian ring.

Some authors have worked on $R[x; \sigma; \delta]$ when R is 2-primal. Recall that a ring R is 2-primal if $N(R) = P(R)$ if and only if $P(R)$ is completely semiprime (i.e. $a^2 \in P(R)$ implies $a \in P(R), a \in R$). We note that any reduced ring is 2-primal, and any commutative ring is also 2-primal. The nature of nil radical, prime ideals, minimal prime ideals, prime radical of $R[x; \sigma; \delta]$ has been investigated, and relations between R and $R[x; \sigma; \delta]$ have been obtained in some cases. For further details on 2-primal rings, we refer the reader to [5, 7, 9, 13].

Recall that in [10], a ring R is called $\sigma$-rigid if $a\sigma(a) = 0$ implies that $a = 0$ for $a \in R$. In [11], a ring R is called a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ and a relation has been established between a $\sigma(*)$-ring and a 2-primal ring. The property is also extended to $R[x; \sigma]$.

Motivated by these developments, in this article, we define a $\delta$-ring (Definition 2), and establish a relation between a $\delta$-ring and a 2-primal ring. We also find a relation between the prime radical of a $\delta$-ring R and that of $R[x; \sigma; \delta]$. We also discuss completely prime ideals and the prime radical of a 2-primal ring R and try to relate completely prime ideals of a ring R with the completely prime ideals of $R[x; \sigma; \delta]$. This is given in Proposition 4. We also find a relation between the prime radical of a 2-primal ring R and that of $R[x; \sigma; \delta]$. This is given in Theorem 4. We generalize this result for a Noetherian $Q$-algebra R. This is given in Corollary 2.

Before this we give a more transparent condition for a Noetherian ring to be a $\sigma(*)$-ring. We prove that if R is a Noetherian ring, then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R. This is proved in Theorem 1. We also prove that if R is a Noetherian $\sigma(*)$-ring, then $P(R[x; \sigma]) = P(R)[x; \sigma]$. This is proved in Theorem 2.
2. Prime Radical of $R[x; \sigma]$ 

Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of a ring $R$. Recall that an ideal $I$ of a ring $R$ is called $\sigma$-invariant if $\sigma(I) = I$ and is called $\delta$-invariant if $\delta(I) \subseteq I$. Also $I$ is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. An ideal $I$ of a ring $R$ is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. With this we have the following definition:

**Definition 1.** (see Kwak [11]) Let $R$ be a ring. Then $R$ is said to be a $\sigma(\ast)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of $R$.

**Proposition 1.** Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$. Then $R$ is a $\sigma(\ast)$-ring implies $P(R)$ is completely semiprime.

**Proof.** Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. \hfill $\square$

**Proposition 2.** Let $R$ be a $\sigma(\ast)$-ring and $U \in \text{Min.Spec}(R)$ be such that $\sigma(U) = U$. Then $US = U[x; \sigma]$ is a completely prime ideal of $S = R[x; \sigma]$.

**Proof.** Proposition 1 implies that $P(R)$ is completely semiprime ideal of $R$ and further more $U$ is completely prime by Proposition 1.11 of [13]. Now we note that $\sigma$ can be extended to an automorphism $\overline{\sigma}$ of $R/U$. Now it is well known that $S/US \simeq (R/U)[x; \overline{\sigma}]$ and hence $US$ is a completely prime ideal of $S$. \hfill $\square$

**Theorem 1.** Let $R$ be a Noetherian ring. Then $R$ is a $\sigma(\ast)$-ring if and only if for each minimal prime $U$ of $R$, $\sigma(U) = U$ and $U$ is completely prime ideal of $R$.

**Proof.** Let $R$ be a Noetherian ring such that for each minimal prime $U$ of $R$, $\sigma(U) = U$ and $U$ is completely prime ideal of $R$. Let $a \in R$ be such that $a\sigma(a) \in P(R) = \cap_{i=1}^n U_i$, where $U_i$ are the minimal primes of $R$. Now for each $i$, $a \in U_i$ or $\sigma(a) \in U_i$ and $U_i$ is completely prime. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence $R$ is a $\sigma(\ast)$-ring.

Conversely, suppose that $R$ is a $\sigma(\ast)$-ring and let $U = U_1$ be a minimal prime ideal of $R$. Now by Proposition 1, $P(R)$ is completely semiprime. Let $U_2, U_3, \ldots, U_n$ be the other minimal primes of $R$. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of $R$. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^n U_i = P(R)$. Therefore
\( a \in P(R) \), and thus \( \cap_{i=1}^{n-1} U_i \subseteq U_n \), which implies that \( U_i \subseteq U_n \) for some \( i \neq n \), which is impossible. Hence \( \sigma(U) = U \).

Now suppose that \( U = U_1 \) is not completely prime. Then there exist \( a, b \in R \setminus U \) with \( ab \in U \). Let \( c \) be any element of \( b(U_2 \cap U_3 \cap \ldots \cap U_n) \). Then \( c^2 \in \cap_{i=1}^{n} U_i = P(R) \). So \( c \in P(R) \) and, thus \( b(U_2 \cap U_3 \cap \ldots \cap U_n) \subseteq U \). Therefore \( bR(U_2 \cap U_3 \cap \ldots \cap U_n)Ra \subseteq U \) and, as \( U \) is prime, \( a \in U, U_i \subseteq U \) for some \( i \neq 1 \) or \( b \in U \). None of these can occur, so \( U \) is completely prime. \( \square \)

**Theorem 2.** Let \( R \) be a Noetherian \( \sigma(\ast) \)-ring. Then \( P(R[x; \sigma]) = P(R)[x; \sigma] \).

**Proof.** Let \( U \in \text{Min.Spec}(R) \). Then \( \sigma(U) = U \) by Theorem 1. Then by (10.6.12) of [12] and by Theorem 7.27 of [4], \( U_1 = U[x, \sigma] \in \text{Min.Spec}(R[x; \sigma]) \). Conversely suppose that \( P \in \text{Min.Spec}(R[x; \sigma]) \). Then \( P \cap R = P \cap R[x; \sigma] \) for some \( J \in \text{Spec}(R) \) and \( J \) contains a minimal prime \( J_1 \). Now \( \sigma(J_1) = J_1 \) by Theorem 1, and then \( P \supseteq J_1[x; \sigma] \), which is a prime ideal of \( R[x; \sigma] \). Therefore \( P = J_1[x; \sigma] \). Hence \( P(R[x; \sigma]) = P(R)[x; \sigma] \). \( \square \)

### 3. Prime Radical of \( R[x; \sigma; \delta] \)

**Definition 2.** Let \( R \) be a ring. Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) be a \( \sigma \)-derivation of \( R \). We say that \( R \) is a \( \delta \)-ring if \( a\delta(a) \in P(R) \) implies \( a \in P(R), a \in R \). We note that a ring \( R \) with identity 1 is not a \( \delta \)-ring as \( \delta(1) = 0 \).

We note that all \( \sigma \)-derivations need not satisfy the property \( (a\delta(a) \in P(R) \) implies \( a \in P(R), a \in R \). For example the following:

Consider \( R = (a_{ij})_{2,2} \), the set of all 2x2 matrices over the ring \( n\mathbb{Z} \), \( n > 1 \) with \( a_{21} = 0 \). Define \( \sigma: R \to R \) by \( \sigma(a_{ij}) = (b_{ij}) \), where \( b_{ij} = a_{ij} \) except that \( b_{12} = -a_{12} \). Then it can be seen that \( \sigma \) is an automorphism of \( R \). Now define \( \delta: R \to R \) by \( \delta(a_{ij}) = (c_{ij}) \), where \( c_{ij} = 0 \) except that \( c_{12} = 2a_{12} + a_{22} - a_{11} \). Then it can be seen that \( \delta \) is a \( \sigma \)-derivation of \( R \). But \( R \) is not a \( \delta \)-ring, as for \( A = (a_{ij})_{2,2} \), with \( a_{ij} = 0 \) except \( a_{22} = 1 \), \( A\delta(A) = (0) \).

**Proposition 3.** Let \( R \) be a 2-primal ring. Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) be a \( \sigma \)-derivation of \( R \) such that \( \delta(P(R)) \subseteq P(R) \). Let \( P \in \text{Min.Spec}(R) \) be such that \( \sigma(P) = P \). Then \( \delta(P) \subseteq P \).

**Proof.** The proof follows from Theorem 3.6 and Lemma 3.2 of [8]. We give a sketch of the proof.

Let \( P \in \text{Min.Spec}(R) \) with \( \sigma(P) = P \). Let \( a \in P \). Then there exists \( b \notin P \).
such that \(ab \in P(R)\) by Corollary 1.10 of [11]. Now we have \(\delta(P(R)) \subseteq P(R)\). Therefore \(\delta(ab) = \delta(a)\sigma(b) + a\sigma(b) \in P(R) \subseteq P\). So we have \(\delta(a)\sigma(b) \in P\). But \(\sigma(b) \notin P\), and therefore \(\delta(a) \in P\) as by Proposition 1.11 of [11] \(P\) is completely prime. Hence \(\delta(P) \subseteq P\).

We now give a relation between a \(\delta\)-ring and a 2-primal ring.

**Theorem 3.** Let \(R\) be a \(\delta\)-ring. Let \(\sigma\) be an automorphism of \(R\) such that \(\sigma(P(R)) = P(R)\), and \(\delta\) be a \(\sigma\)-derivation of \(R\) such that \(\delta(P(R)) \subseteq P(R)\). Then \(R\) is 2-primal.

**Proof.** Define a map \(\partial : R/P(R) \to R/P(R)\) by \(\partial(a + P(R)) = \delta(a) + P(R)\) for \(a \in R\) and \(\tau : R/P(R) \to R/P(R)\) a map by \(\tau(a + P(R)) = \sigma(a) + P(R)\) for \(a \in R\). Now it is easy to see that \(\tau\) is an automorphism of \(R/P(R)\). Also for any \(a + P(R), b + P(R) \in R/P(R)\); \(\partial((a + P(R))(b + P(R))) = \partial(ab + P(R)) = \delta(ab) + P(R) = \delta(a)\sigma(b) + a\delta(b) + P(R) = (\delta(a) + P(R))\sigma(b) + P(R) + (a + P(R))(\delta(b) + P(R)) = \delta(a + P(R))\tau(b + P(R)) + (a + P(R))\partial(b + P(R))\), and it is obvious that \(\partial(a + P(R) + b + P(R)) = \partial(a + P(R)) + \partial(b + P(R))\). Therefore \(\partial\) is a \(\tau\) - derivation of \(R/P(R)\). Now a \(\delta(a) \in P(R)\) if and only if \((a + P(R))\partial(a + P(R)) = P(R)\) in \(R/P(R)\). Thus, as in Proposition 5 of [6], \(R\) is a reduced ring and hence \(R\) is 2-primal.

We notice that a 2-primal ring need not be a \(\delta\)-ring, as can be seen from the following example.

Consider \(R = Z_2 \oplus Z_2\). Then \(R\) is a commutative reduced ring, and so is a 2-primal ring. Define a map \(\sigma : R \to R\) by \(\sigma(a, b) = (b, a)\). Then \(\sigma\) is an automorphism of \(R\). Now define a map \(\delta : R \to R\) by \(\delta(a, b) = (a-b, 0)\). Then \(\delta\) is a \(\sigma\)-derivation of \(R\). But \(R\) is not a \(\delta\)-ring, as \((0, 1)\delta(0,1) = (0, 0)\).

**Proposition 4.** Let \(R\) be a ring. Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) be a \(\sigma\)-derivation of \(R\). Then:

1. For any completely prime ideal \(P\) of \(R\) with \(\delta(P) \subseteq P\) and \(\sigma(P) = P\), \(P[x; \sigma; \delta]\) is a completely prime ideal of \(R[x; \sigma; \delta]\).
2. For any completely prime ideal \(U\) of \(R[x; \sigma; \delta]\), \(U \cap R\) is a completely prime ideal of \(R\).

**Proof.** (1) Proposition 3 implies that \(P(R)\) is completely semiprime ideal of \(R\). Now we note that \(\sigma\) can be extended to an automorphism \(\overline{\sigma}\) of \(R/P\), and \(\delta\) can be extended to a \(\sigma\)-derivation \(\overline{\delta}\) of \(R/P\). Now it is well known that \(R[x; \sigma; \overline{\delta}]/P[x; \sigma; \overline{\delta}] \simeq (R/P)[x; \overline{\sigma}; \overline{\delta}]\) and hence \(P[x; \sigma; \delta]\) is a completely prime ideal of \(R[x; \sigma; \delta]\).

2. Let \(U\) be a completely prime ideal of \(R[x; \sigma; \delta]\). Suppose \(a, b \in R\) are
such that $ab \in U \cap R$ with $a \notin U \cap R$. This means that $a \notin U$ as $a \in R$. Thus we have $ab \in U \cap R \subseteq U$, with $a \notin U$. Therefore we have $b \in U$, and thus $b \in U \cap R$.

The above discussion leads to the following question:

Is $\delta(U \cap R) \subseteq U \cap R$ in Proposition 4? If so, is $U = \delta(U \cap R)$? The question remains to be answered, but in this connection we note that $\sigma$ and $\delta$ can be extended to $R[x; \sigma; \delta]$ by taking $\sigma(x) = x$ and $\delta(x) = 0$. In other words, $\sigma(xa) = x\sigma(a)$ and $\delta(xa) = x\delta(a)$ for all $a \in R$.

**Corollary 1.** Let $R$ be a $\delta$-ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$ such that $\delta(P(R)) \subseteq P(R)$. Let $P \in \text{Min.Spec}(R)$ be such that $\sigma(P) = P$. Then $P[x; \sigma; \delta]$ is a completely prime ideal of $R[x; \sigma; \delta]$.

**Proof.** $R$ is 2-primal by Theorem 3, and so by Proposition 3 $\delta(P) \subseteq P$. Further more $P$ is a completely prime ideal of $R$ by Proposition 1.11 of [11]. Now use Proposition 4.

**Theorem 4.** Let $R$ be a $\delta$-ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$ such that $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in \text{Min.Spec}(R)$. Then $R[x; \sigma; \delta]$ is 2-primal if and only if $P(R)[x; \sigma; \delta] = P(R[x; \sigma; \delta])$.

**Proof.** Let $R[x; \sigma; \delta]$ be 2-primal. Let $P \in \text{Min.Spec}(R)$. By Corollary 1 $P[x; \sigma; \delta]$ is a completely prime ideal of $R[x; \sigma; \delta]$, and therefore $P(R[x; \sigma; \delta]) \subseteq P(R)[x; \sigma; \delta]$. One may see Proposition 3.8 of [8] also. Let $f(x) = \sum x^i a_j \in P(R)[x; \sigma; \delta]$, $0 \leq i \leq n$. Now $R$ is a 2-primal sub ring of $R[x; \sigma; \delta]$ by Theorem 3. This implies that $a_j$ is nilpotent and thus $a_j \in N(R[x; \sigma; \delta]) = P(R[x; \sigma; \delta])$, and so we have $x^i a_j \in P(R[x; \sigma; \delta])$ for each $j$. Therefore $f(x) \in P(R[x; \sigma; \delta])$. Hence we have $P(R)[x; \sigma, \delta] = P(R[x; \sigma; \delta])$.

Conversely suppose $P(R)[x; \sigma; \delta] = P(R[x; \sigma; \delta])$. We will show that $R[x; \sigma; \delta]$ is 2-primal. Let $g(x) = \sum x^i b_i \in R[x; \sigma; \delta]$, $0 \leq i \leq n$ be such that $(g(x))^2 \in P(R[x; \sigma; \delta]) = P(R)[x; \sigma; \delta]$. Then by an easy induction and by using the fact that $P(R)$ is completely semiprime by Theorem 3, it can be easily seen that $b_i \in P(R)$ for all $b_i$, $0 \leq i \leq n$. This means that $f(x) \in P(R)[x; \sigma; \delta] = P(R[x; \sigma; \delta])$. Therefore $P(R[x; \sigma; \delta])$ is completely semiprime. Hence $R[x; \sigma; \delta]$ is 2-primal.

We now generalize the above result for a Noetherian $Q$-algebra $R$, and towards this we have the following:

**Proposition 5.** Let $R$ be a Noetherian $Q$-algebra. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for $a \in R$. 
Then:

(1) \(\sigma(N(R)) = N(R)\).

(2) If \(P \in \text{Min.Spec}(R)\) such that \(\sigma(P) = P\), then \(\delta(P) \subseteq P\).

Proof. (1) Denote \(N(R)\) by \(N\). We have \(\sigma(N) \subseteq N\) as \(\sigma(N)\) is a nilpotent ideal of \(R\). Now for any \(n \in N\), there exists \(a \in R\) such that \(n = \sigma(a)\). So \(I = \sigma^{-1}(N) = \{a \in R \mid \sigma(a) = n \in N\}\) is an ideal of \(R\). Now \(I\) is nilpotent, therefore \(I \subseteq N\), which implies that \(N \subseteq \sigma(N)\). Hence \(\sigma(N) = N\).

(2) Let \(T = \{a \in P \mid \delta^k(a) \in P\ \text{for all integers } k \geq 1\}\). Then \(T\) is a \(\delta\)-invariant ideal of \(R\). Now it can be seen that \(T \in \text{Spec}(R)\), and since \(P \in \text{Min.Spec}(R)\), we have \(T = P\). Hence \(\delta(P) \subseteq P\).

Corollary 2. Let \(R\) be a Noetherian \(\delta\)-ring, which is also an algebra over \(Q\). Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) be a \(\sigma\)-derivation of \(R\) such that \(\sigma(\delta(a)) = \delta(\sigma(a))\), for \(a \in R\). Let \(\sigma(P) = P\) for all \(P \in \text{Min.Spec}(R)\). Then \(R[x; \sigma; \delta]\) is 2-primal if and only if \(P(R)[x; \sigma; \delta] = P(R[x; \sigma; \delta])\).

Proof. Use Theorem 4 and Proposition 5.

Theorem 5. Let \(R\) be a Noetherian \(Q\)-algebra. Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) be a \(\sigma\)-derivation of \(R\) such that \(\sigma(\delta(a)) = \delta(\sigma(a))\), for \(a \in R\). Let \(R\) be a \(\sigma(\ast)\)-ring and a \(\delta\)-ring. Then \(P(R)[x; \sigma; \delta] = P(R[x; \sigma; \delta])\).

Proof. Let \(U \in \text{Min.Spec}(R)\). Then Theorem 1 implies that \(\sigma(U) = U\), and now Proposition 5 implies that \(\delta(U) \subseteq U\). Now on the same lines as in (2) we get \(P(R)[x; \sigma; \delta] = P(R[x; \sigma; \delta])\).

Corollary 3. Let \(R\) be as in Theorem 5. Then \(R[x; \sigma; \delta]\) is 2-primal.

Proof. Theorem 1 implies that \(\sigma(P) = P\), and Proposition 5 implies that \(\delta(P) \subseteq P\) for all \(P \in \text{Min.Spec}(R)\). Now Theorem 5 implies that \(P(R)[x; \sigma; \delta] = P(R[x; \sigma; \delta])\). Now use Theorem 4.

References


