A POLYNOMIAL TIME ALGORITHM FOR MINIMIZING A NONDECREASING SUPERMODULAR SET FUNCTION AND ITS PERFORMANCE GUARANTEE

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Abstract: This paper presents a polynomial algorithm for minimizing a non-decreasing supermodular set function and discusses its performance guarantee.

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1. Introduction and Concepts

Let $I = \{1, 2, \ldots, n\}$, $\Omega = \{x/ x \subseteq I\}$ and $f : \Omega \rightarrow R$ is a supermodular set function, i.e. for all $X, Y \subseteq I$

$$f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y).$$

We consider the following problem:

$$\min \{f(X) : X \in \Omega, |x| = p\},$$

(1)

where $\Omega$ is a finite set; $p$ is a positive integer, $p < n$.

The well-known p-median problem is a special case of (1) with matrix with the row index set $I$ and the column index set $J$.

The problem of minimization a supermodular set function on comatroid
whose special case is the well-known NP-hard minimization p-median problem. In all of those approximation algorithm to solve combinatorial optimization, greedy algorithm and approximation algorithm is simpler and effective algorithm. Ilev and Linker [2], [3] give a greedy algorithm to solve problem (1) and analysis the performance guarantee. The main result of this paper is a tight bound on the performance guarantee of a Local Search Algorithm for nonnegative nondecreasing supermodular functions $f(x)$.

The reminder of the paper is organized as follows. In Section 2 we give some propositions and their proofs. In Section 3 we give an approximation algorithm for minimizing a nondecreasing supermodular set function and its performance guarantee.

Let us denote $X \cup \{y\}$ by $X + \{y\}; X \setminus \{y\}$ by $X - \{x\}$ for all $X \subseteq I, x \in X$ and $y \in I \setminus X$.

### 2. Some Properties of Supermodular Set Functions

Let $I$ be a finite set $f : \Omega \rightarrow R$ be a nonnegative nondecreasing supermodular set function. For $X \subseteq I$ and $x \in X$ we set $d_x(x) = f(x) - f(x - \{x\}) \geq 0$.

**Lemma 1.** $d_x(X) \leq d_x(Y)$ for all $X, Y \subseteq I, X \subseteq Y$ and $x \in X$ and $f : \Omega \rightarrow R$ is nondecreasing supermodular set functions.

**Proof.** It follows from the supermodular of $f$ that

$$d_x(X) = f(X) - f(X - \{x\}) = f(X) - f((Y - \{x\}) \cap X)$$

$$\leq f(X) + f(Y) - f(X) - f(Y - \{x\}) = f(Y) - f(Y - \{x\}) = d_x(Y). \quad \square$$

We consider the following parameter of a nondecreasing supermodular set function $f$

$$\theta = \max_{d_x(I) > 0, x \in I} \frac{d_x(I) - d_x(\{x\})}{d_x(I)},$$

and call $\theta$ a steepness of a set function $f$. It is easy to see that $\theta \in [0, 1]$.

**Lemma 2.** $(1 - \theta)d_x(I) \leq d_x(\{x\})$ for all $x \in I$.

**Proof.** It follows from the definition of steepness that

$$\theta \geq \frac{d_x(I) - d_x(\{x\})}{d_x(I)},$$

for each $d_x(I) > 0$. Hence $(1 - \theta)d_x(I) \leq d_x(\{x\})$. \quad \square

If $d_x(I) = 0$, the inequality evidently holds.
Let $A = \{a_1, a_2, ..., a_m\} \subseteq I$, $B = \{b_1, b_2, ..., b_n\} \subseteq I$. Set $A_0 = \emptyset$, $A_j = \{a_1, a_2, ..., a_j\}$ ($j = 1, 2, ..., m$), $A = A_m$, $B_0 = \emptyset$, $B_j = \{b_1, b_2, ..., b_j\}$ ($j = 1, 2, ..., n$), $B = B_n$.

**Lemma 3.**

$$f(A \cup B) = f(B) + \sum_{a_j \in A \setminus B} d_{a_j}(B \cup A_j) = f(A) + \sum_{b_j \in B \setminus A} d_{b_j}(A \cup B_j).$$

**Proof.** We prove the first equality

$$f(A \cup B) = f(B \cup A_m) = f((B \cup A_{m-1}) \cup \{a_m\}) = \begin{cases} f(B \cup A_{m-1}), & a_m \in B, \\ f(B \cup A_{m-1}) + d_{a_m}(B \cup A_m), & a_m \in A \setminus B. \end{cases}$$

Similarly

$$f(B \cup A_{m-1}) = \begin{cases} f(B \cup A_{m-2}), & a_{m-1} \in B, \\ f(B \cup A_{m-2}) + d_{a_{m-1}}(B \cup A_{m-1}), & a_{m-1} \in A \setminus B, \end{cases}$$

and so on. Finally, we obtain

$$f(A \cup B) = f(B \cup A_0) + \sum_{a_j \in A \setminus B} d_{a_j}(B \cup A_j) = f(B) + \sum_{a_j \in A \setminus B} d_{a_j}(B \cup A_j).$$

The second equality can be proved similarly. \(\square\)

**Lemma 4.** $f(B) \geq \sum_{b_j \in B \setminus A} d_{b_j}(B_j)$ for $f$ be a nonnegative nondecreasing supermodular set function.

**Proof.** Since $f(B_0) = f(\emptyset) \geq 0$, then

$$f(B) = f(B_n) - f(B_{n-1}) + f(B_{n-1}) - f(B_{n-2}) + f(B_{n-2}) + \ldots + f(B_1) - f(B_1) + f(B_0) \geq d_{b_n}(B_n) + d_{b_{n-1}}(B_{n-1}) + \ldots + d_{b_1}(B_1) = \sum_{b_j \in B} d_{b_j}(B_j).$$

Since $f : \Omega \rightarrow R$ is a nondecreasing set function we have $d_{b_j}(B_j) \geq 0$ for all $b_j \in B_j$. So, we obtain $f(B) \geq \sum_{b_j \in B} d_{b_j}(B_j) \geq \sum_{b_j \in B \setminus A} d_{b_j}(B_j)$

This completes the proof. \(\square\)

**Lemma 5.**

$$(1 - \theta + \frac{\theta}{1 - \theta})f(B) - (1 - \theta)f(A) \geq \sum_{i=1}^{k} \left[ f(A + \{b_i\} - \{a_i\}) - f(A) \right],$$

for $|A| = |B| = P$, $|A \setminus B| = |B \setminus A| = k$ and for $A \setminus B = \{a_{i_1}, a_{i_2}, ..., a_{i_k}\}$, $B \setminus A = \{b_{j_1}, b_{j_2}, ..., b_{j_k}\}$. 


Proof. By Lemma 3
\[(1 - \theta)f(B) - (1 - \theta)f(A)\]
\[= (1 - \theta) \sum_{b_j \in B \setminus A} d_{b_j}(A \cup B_j) - (1 - \theta) \sum_{a_j \in A \setminus B} d_{a_j}(B \cup A_j).\] (4)

For each \(a_j \in A \setminus B\) (see Lemmas 1 and 2)
\[(1 - \theta)d_{a_j}(B \cup A_j) \leq (1 - \theta)d_{a_j}(I) \leq d_{a_j}(\{a_j\}) \leq d_{a_j}(A + \{b_j\}).\]
Therefore
\[(1 - \theta) \sum_{a_j \in A \setminus B} d_{a_j}(B \cup A_j) \leq \sum_{a_j \in A \setminus B} d_{a_j}(A + \{b_j\}).\]

By Lemma 1, for each \(b_j \in B \setminus A\)
\[d_{b_j}(A \cup B_j) \geq d_{b_j}(A + \{b_j\}).\]

By (4):
\[(1 - \theta)[f(B) - f(A)] \geq (1 - \theta) \sum_{b_j \in B \setminus A} d_{b_j}(A + \{b_j\}) - \sum_{a_j \in A \setminus B} d_{a_j}(A + \{b_j\})\]
\[= \sum_{b_j \in B \setminus A} d_{b_j}(A + \{b_j\}) - \sum_{a_j \in A \setminus B} d_{a_j}(A + \{b_j\}) - \theta \sum_{b_j \in B \setminus A} d_{b_j}(A + \{b_j\}).\] (5)

By Lemma 1 and 2:
\[(1 - \theta)d_{b_j}(A + \{b_j\}) \leq (1 - \theta)d_{b_j}(I) \leq d_{b_j}(\{b_j\}) \leq d_{b_j}(B_j).\]

By Lemma 4:
\[\sum_{b_j \in B \setminus A} d_{b_j}(A + \{b_j\}) \leq \frac{1}{1 - \theta} \sum_{b_j \in B \setminus A} d_{b_j}(B_j) \leq \frac{1}{1 - \theta} f(B).\]

By (5):
\[(1 - \theta)[f(B) - f(A)] \geq \sum_{b_j \in B \setminus A} d_{b_j}(A + \{b_j\}) - \sum_{a_j \in A \setminus B} d_{a_j}(A + \{b_j\}) - \frac{\theta}{1 - \theta} f(B).\]

Hence:
\[(1 - \theta + \frac{\theta}{1 - \theta})f(B) - (1 - \theta)f(A) \geq \sum_{b_j \in B \setminus A} d_{b_j}(A + \{b_j\}) - \sum_{a_j \in A \setminus B} d_{a_j}(A + \{b_j\})\]
\[= \sum_{t=1}^{k} [f(A + \{b_{j_t}\}) - f(A)] - \sum_{t=1}^{k} [f(A + \{b_{j_t}\}) - f(A + \{b_{j_t}\}) - a_{j_t}].\]
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\[ k \sum_{t=1}^{k} [f(A + \{b_{ji}\}) - \{a_{ji}\} - f(A)], \]

as required. □

3. Approximation Algorithm and its Performance Guarantee

We now describe the approximation algorithm for solving the problem (1):

Step i: Set \( 0 < \varepsilon < 1 \) select \( x_0 \subseteq X \) such that \(|x_0| = P\); let \( i = 0 \).

Step ii: Search \( u \in I \setminus X_i \) and \( v \in X_i \) such that \( f(X_i + \{u\} - \{v\}) = \min_{x \in I \setminus X_i, y \in X_i} f(X_i + \{x\} - \{y\}) \).

Step iii: Set \( X_{i+1} = X_i + \{u\} + \{v\} \), if \( f(X_{i+1}) - f(X_i) \geq -\frac{(1-\theta)\varepsilon}{n} f(X_i) \) then stop; otherwise let \( i = i + 1 \) go to Step ii.

Let \( x \subseteq I \) satisfy \(|x| = P\), we call \( N(x) = \{Y \subseteq I/|Y| = 2, |Y - X| = 2\} \) is the field of \( X \), from the above steps we can see that the above algorithm always get its new minimization approximation solution set of optimal function in current approximate solution set in the field.

Performance guarantees of algorithm.

Theorem 1. Let \( X^* \) be optimal solution of problem (1); \( \bar{X} \) be the solution returned by approximation algorithm, then

\[ f(\bar{X}) \leq \frac{1}{1 - \varepsilon} [1 + \frac{\theta}{(1 - \theta)^2}] f(X^*). \]

Proof. Let \( X^* = \{x_1^*, x_2^*, ..., x_p^*\} \), \( \bar{X} = \{\bar{x}_1, \bar{x}_2, ..., \bar{x}_p\} \). We assume \(|X^* \setminus \bar{X}| = |\bar{X} \setminus X^*| = K \) and \( X^* \setminus \bar{X} = \{x_{i_1}^*, x_{i_2}^*, ..., x_{i_k}^*\}, \bar{X} \setminus X^* = \{\bar{x}_{j_1}, \bar{x}_{j_2}, ..., \bar{x}_{j_k}\} \).

By Lemma 5, for \( B = X^* \), and \( A = \bar{X} \). We can obtain

\[ (1 - \theta + \frac{\theta}{1 - \theta})f(X^*) - (1 - \theta)f(\bar{X}) \geq \sum_{t=1}^{k} [f(\bar{x} + \{x_{i_t}^*\} - \{\bar{x}_{j_t}\}) - f(\bar{x})]. \]

By (6) for every \( t \) satisfy \( 1 \leq t \leq l \) we have

\[ f(\bar{x} + \{x_{i_t}^*\} - \{\bar{x}_{j_t}\}) - f(\bar{x}) \geq -\frac{(1-\theta)\varepsilon}{n} f(\bar{X}). \]

Hence

\[ (1 - \theta + \frac{\theta}{1 - \theta})f(X^*) - (1 - \theta)f(\bar{X}) \geq -\frac{k(1-\theta)\varepsilon}{n} f(\bar{X}) \geq -(1 - \theta)\varepsilon f(\bar{X}). \]
Therefore

\[ f(\tilde{X}) \leq \frac{1}{1 - \varepsilon}[1 + \frac{\theta}{(1 - \theta)^2}]f(X^*) . \]

This completes the proof. \( \square \)

**Theorem 2.** The above algorithm can get the approximation solution of Theorem 1 at the most \( o(n^3) \) iteratives.

**Proof.** Let \( X_0, X_1, X_2, ..., X_l = \tilde{X} \) be the approximal solution sequence of the above algorithm of problem (1).

By (6) for \( i = 0, 1, 2, ..., l - 1 \) we have

\[ f(X_{i+1}) - f(X_i) \leq -\frac{(1 - \theta)\varepsilon}{n} f(X_i) . \]

That is

\[ f(X_{i+1}) \leq [1 - \frac{(1 - \theta)\varepsilon}{n}]f(X_i), \quad i = 0, 1, 2, ..., l - 1 . \]

Hence

\[ f(X^*) \leq f(\tilde{X}) = f(X_k) \leq [1 - \frac{(1 - \theta)\varepsilon}{n}]f(X_0) . \]

Therefore

\[ l \log[1 - \frac{(1 - \theta)\varepsilon}{n}] \geq \log \frac{f(X^*)}{f(X_0)} . \]

By \( \log[1 - \frac{(1 - \theta)\varepsilon}{n}] \leq -\frac{(1 - \theta)\varepsilon}{n} \) we have \( k \leq \frac{n}{(1 - \theta)\varepsilon} \log \frac{f(X_0)}{f(X^*)} \).

This concludes the proof. \( \square \)

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**References**

