International Journal of Pure and Applied Mathematics
Volume 53 No. 1 2009, 21-36

ON THE GLOBAL CHARACTER OF THE SYSTEM $x_{n+1} = \frac{\alpha_1+\gamma_1 y_n}{x_n}$ and $y_{n+1} = \frac{\beta_2 x_n+\gamma_2 y_n}{B_2 x_n+C_2 y_n}$

E. Camouzis[1], G. Ladas[2 §], L. Wu[3]

[1]Department of Mathematics
American College of Greece
6, Gravias Street, Aghia Paraskevi, 15342, Athens, GREECE
e-mail: camouzis@acgmail.gr

[2,3]Department of Mathematics
University of Rhode Island
Kingston, RI 02881-0816, USA
[2]e-mail: gladas@math.uri.edu
[3]e-mail: liwu@math.uri.edu

Abstract: We investigate the global character of solutions of the rational system in the title with nonnegative parameters and with arbitrary positive initial conditions. In particular we obtain necessary and sufficient conditions for all solutions of the system to be bounded. We also show that in a certain region of the parameters every solution of the system converges to a finite limit or to a period-two solution.

AMS Subject Classification: 39A10
Key Words: boundedness, convergence, period-two convergence, rational systems, stability

1. Introduction

We investigate the global character of solutions of the rational system in the plane,

Received: March 16, 2009

[§]Correspondence author

$$\left.\begin{array}{l} x_{n+1} = \dfrac{\alpha_1 + \gamma_1 y_n}{x_n} \\ y_{n+1} = \dfrac{\beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n} \end{array}\right\}, \quad n = 0, 1, \ldots \tag{1}$$

with nonnegative parameters and with arbitrary positive initial conditions. In particular we obtain necessary and sufficient conditions for all solutions of the system to be bounded. We also show that in a certain region of the parameters every solution of system (1) converges to a finite limit or to a period-two solution.

In the numbering system which was introduced in [6], system (1), with positive parameters, is the special case, $\#(23, 36)$. If we allow one or more of the parameters of system (1) to be zero, then system (1) contains 27 special cases of systems with positive parameters. The first equation of the system is any of the three equations in Table 1, of Appendix 1, and the second equation is any equation of the nine equations in Table 2, of Appendix 1. Note that the equations are written in normalized form and we use the numbering system which was introduced in [6].

Our main result about the boundedness character of solutions of system (1) is that when

$$\gamma_1 > 0,$$

every solution $\{x_n,\ y_n\}$ is bounded if and only if

$$C_2\beta_2 \le B_2\gamma_2.$$

In Section 2, we present the global character of solutions of system (1) in the special case where

$$\gamma_1 = 0.$$

This accounts for the first 9 of the 27 special cases contained in system (1), see Appendix II.

In Section 3, we present the boundedness character of solutions of system (1) in the special case where

$$\alpha_1 = 0.$$

This accounts for another 9 of the 27 special cases which are contained in system (1), see Appendix II.

In Section 4, we present the boundedness character of solutions of system (1) in the special case where

$$\alpha_1,\ \gamma_1\ \in (0,\ \infty).$$

This account for the last 9 of the 27 special cases contained in system (1), see Appendix II.

It is interesting to note that the methods and techniques that have been developed to understand the character of solutions of single rational difference equations also apply here in the investigation of system (1), see [1]-[4], [7], [18], [22], [23] and [26]. For some work on rational systems see [5], [6], [9]-[13], and [24], [25].

The quotient

$$z_n = \frac{x_n}{y_n}, \quad \text{for } \; n \geq 0 \tag{2}$$

of the two components x_n and y_n of each solution $\{x_n, y_n\}$ of system (1) satisfies the single second-order rational difference equation

$$z_{n+1} = \frac{(B_2 z_n + C_2)[(\alpha_1 B_2 + \gamma_1 \beta_2) z_{n-1} + (\alpha_1 C_2 + \gamma_1 \gamma_2)]}{z_n(\beta_2 z_n + \gamma_2)(\beta_2 z_{n-1} + \gamma_2)}, \quad n = 0, 1, \ldots . \tag{3}$$

It is interesting to note that the function

$$f(u,v) = \frac{(B_2 u + C_2)[(\alpha_1 B_2 + \gamma_1 \beta_2) v + (\alpha_1 C_2 + \gamma_1 \gamma_2)]}{u(\beta_2 u + \gamma_2)(\beta_2 v + \gamma_2)}$$

which is involved in equation (3), decreases in both variables when

$$C_2 \beta_2 > B_2 \gamma_2$$

and when

$$C_2 \beta_2 < B_2 \gamma_2$$

decreases in u and increases in v. In view of the monotonicity of the function f the following theorem will be useful in our understanding of the global character of solutions of equation (3) and then in turn for system (1).

Theorem 1.1. (see Camouzis and Ladas, [7], p. 11, or [8]) *Let I be a set of real numbers and let*

$$F: \; I \times I \to I$$

be a function $F(u,v)$, which decreases in u and increases in v. Then for every solution $\{z_n\}_{n=-1}^{\infty}$ of the equation

$$z_{n+1} = F(z_n, z_{n-1}), \quad n = 0, 1, \ldots \tag{4}$$

the subsequences $\{z_{2n}\}_{n=0}^{\infty}$ and $\{z_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms are eventually monotonic.

The change of variables (2), which led us from system (1) to equation (3) can be profitably employed in any system where one of the equations is homogeneous. For example applying this idea to the system

$$(12,18): \quad \left.\begin{aligned} x_{n+1} &= \frac{\alpha_1}{x_n + y_n} \\ y_{n+1} &= \frac{\beta_2 x_n}{x_n + y_n} \end{aligned}\right\}, \qquad \text{for } \; n = 0, 1, \ldots, \tag{5}$$

we find that the quotient satisfies the rational difference equation

$$z_{n+1} = \frac{\alpha_1(1 + z_{n-1})}{\beta_2 z_n z_{n-1}}, \;\; n = 0, 1, \ldots \; .$$

This equation was investigated in [3] and it was shown that every solution has a positive limit. In particular, this implies that every solution of system (5) converges to a finite limit.

For some basic results related to the local stability of difference equations see [7], [14], [21], and [26].

2. The Special Case $\gamma_1 = 0$

This section addresses the first 9 special cases in Appendix II, where

$$\gamma_1 = 0.$$

In each of the 2 special cases

$$(2,5) \quad \text{and} \quad (2,9),$$

of system (1), every solution of the system is clearly periodic with period 2.

In the special case $(2,6)$, we easily see that the component $\{x_n\}$ of every solution $\{x_n,\ y_n\}$ is periodic with period two, while for the $\{y_n\}$ component, we find,

$$y_{2n} = \gamma_2^{2n} y_0, \qquad \text{and} \qquad y_{2n+1} = \gamma_2^{2n+1} \frac{y_0}{x_0}, \qquad \text{for } \; n \geq 0.$$

Therefore, every solution in this case converges to a period-two solution if and only if

$$\gamma_2 \leq 1,$$

and when

$$\gamma_2 > 1,$$

the $\{y_n\}$ component is unbounded and diverges to ∞, see Appendix II.

In the special case $(2,8)$, we easily see that $\{x_n\}$ is periodic with period two, while for the $\{y_n\}$ component we find,

$$y_{2n} = \frac{y_0}{x_0^{2n}} \qquad \text{and} \qquad y_{2n+1} = \frac{\gamma_2 x_0^{2n+1}}{y_0}, \qquad \text{for } \ n \geq 0.$$

Therefore, every solution in this case converges to a period-two solution if and only if

$$x_0 = 1,$$

and when

$$x_0 \neq 1$$

the $\{y_n\}$ component is unbounded with the subsequences of even and odd terms converging one of them to zero and the other to ∞.

In each of the following 4 special cases of system (1):

$$(2,15), \quad (2,18), \quad (2,26), \quad \text{and} \quad (2,36)$$

the $\{x_n\}$ component of the solution is periodic with period two, while the $\{y_n\}$ component satisfies a Riccati difference equation with period-two coefficients. It follows from known results that the component $\{y_n\}$ converges to a periodic-two sequence, see [17]. Therefore in each of these 4 special cases, every solution of each system is bounded and converges to a period-two solution of the system, see Appendix II.

Finally, in the special case

$$(2,27),$$

the $\{x_n\}$ component of the solution is periodic with period two while the subsequences of the even and odd terms of the $\{y_n\}$ component satisfy the system:

$$\left.\begin{array}{l} y_{2n+2} = \beta_2(1+\gamma_2 x_0) + \gamma_2^2 y_{2n} \\ y_{2n+1} = \beta_2(1+\frac{\gamma_2}{x_0}) + \gamma_2^2 y_{2n-1} \end{array}\right\}, \quad n = 0, 1, \cdots$$

From this it follows that every solution in this special case converges to a period two solution when

$$\gamma_2 < 1$$

and when $\gamma_2 \geq 1$, the $\{y_n\}$ component converges to ∞, see Appendix II.

3. The Special Case $\alpha_1 = 0$

This section addresses the 9 special cases in the middle of Appendix II where

$$\alpha_1 = 0.$$

In each of the 2 special cases

$$(8,5) \quad \text{and} \quad (8,9),$$

every solution of the system is clearly periodic with period 2.

In the special case

$$(8,6),$$

every solution is constant, for $n \geq 2$, and so every solution is bounded and converges to the equilibrium point $(1,1)$.

In the special case

$$(8,8),$$

the solution of the system is given explicitly as follows:

$$x_n = (\frac{x_0}{y_0})^n, \qquad \text{and} \qquad y_n = (\frac{y_0}{x_0})^n, \qquad \text{for} \;\; n \geq 1.$$

Therefore we have unbounded solutions if and only if

$$x_0 \neq y_0.$$

In each of the special cases,

$$(8,15) \quad \text{and} \quad (8,27),$$

the $\{x_n\}$ component of the solution satisfies a Riccati difference equation from which it follows that in each of these two cases every solution is bounded and converges to a positive limit.

Next we will address the character of solutions of the last three systems in this section, namely:

$$(8,18), \quad (8,26), \quad \text{and} \quad (8,36).$$

In normalized form the above three systems are included in the system,

$$\left.\begin{aligned} x_{n+1} &= \frac{y_n}{x_n}, \\ y_{n+1} &= \frac{x_n + \gamma_2 y_n}{B_2 x_n + y_n}, \end{aligned}\right\} \qquad n = 0, 1, \dots, \tag{6}$$

with nonnegative parameters γ_2 and B_2.

Note that

$$y_n = x_{n+1}\, x_n,$$

and so from the second equation of the system we find that $\{x_n\}$ satisfies the first order nonlinear equation,

$$x_{n+1} = \frac{\gamma_2 x_n + 1}{(x_n + B_2) x_n}, \qquad n = 0, 1, \dots. \tag{7}$$

The equilibrium point $\bar{x}$ of equation (7) satisfies

$$\bar{x}^2(B_2+\bar{x}) = 1+\gamma_2\bar{x}. \tag{8}$$

Set $q(x) = \frac{1+\gamma_2 x}{x(B_2+x)}$, and note that $q(x)$ is a decreasing function.

Lemma 3.1. *The following statements are true for the solutions of equation (7):*

When $B_2\gamma_2 > 1$, all solutions converge to the equilibrium $\bar{x}$.

When $B_2\gamma_2 = 1$, all nontrivial solutions are periodic with prime period two.

When $B_2\gamma_2 < 1$, all nontrivial solutions are unbounded.

Proof. The proof follows directly from the fact that

$$q(q(x)) - x = x\cdot(1-B_2\gamma_2)\cdot\frac{x^3+B_2x^2-\gamma_2x-1}{1+B_2x^2+B_2^2x+\gamma_2x}. \qquad \square$$

4. The Special Case $\alpha_1,\ \gamma_1 \in\ (0,\infty)$

This section addresses the last 9 special cases in Appendix II where

$$\alpha_1,\ \gamma_1 \in (0,\infty).$$

In the special cases,

$$(23,5), \quad \text{and} \quad (23,9),$$

every solution $\{x_n, y_n\}$ of the system is periodic with period 2.

In the special case

$$(23,6),$$

the component $\{y_n\}$ of the solution satisfies a Riccati equation and so $\{y_n\}$ converges to a finite limit. From the system it is now easily seen that $\{x_n\}$ also converges to a finite limit.

Next we investigate the global character of solutions of the special case $(23,8)$ which in normalized form is the system:

$$(23,8): \qquad \left.\begin{aligned} x_{n+1} &= \frac{\alpha_1+y_n}{x_n}, \\ y_{n+1} &= \frac{x_n}{y_n}, \end{aligned}\right\}, \qquad n=0,1,\ldots. \tag{9}$$

From the second equation in (9) we see that

$$x_n = y_{n+1}\ y_n,$$

and so from the first equation we find that

$$y_{n+1} = \frac{\alpha_1 + y_{n-1}}{y_{n-1}y_n^2}, \qquad n = 0, 1, \ldots.$$

The equilibrium points of system (9) are the points $(\bar{x}, \bar{y})$ which satisfy

$$\bar{x} = \frac{\alpha_1 + \bar{y}}{\bar{x}}, \qquad \bar{y} = \frac{\bar{x}}{\bar{y}}.$$

Here we show that system (9) has unbounded solutions.

Lemma 4.1. *system (9) has unbounded solutions. More specifically, for any initial conditions* (x_0, y_0) *such that*

$$(x_0 - \bar{x})(y_0 - \bar{y}) < 0,$$

either

$$\lim_{n\to\infty} x_{2n} = \infty, \quad \lim_{n\to\infty} x_{2n-1} = 0, \quad \lim_{n\to\infty} y_{2n} = 0, \quad \lim_{n\to\infty} y_{2n-1} = \infty,$$

or

$$\lim_{n\to\infty} x_{2n} = 0, \quad \lim_{n\to\infty} x_{2n-1} = \infty, \quad \lim_{n\to\infty} y_{2n} = \infty, \quad \lim_{n\to\infty} y_{2n-1} = 0.$$

Furthermore, every bounded solution of system (9) converges to a finite limit.

Proof. Let $\{x_n, y_n\}$ be a positive solution of system (9). We will give the proof when $x_0 < \bar{x}$ and $y_0 > \bar{y}$. The proof in the other case is similar and will be omitted. Then

$$x_1 = \frac{\alpha_1 + y_0}{x_0} > \frac{\alpha_1 + \bar{y}}{\bar{x}} = \bar{x} \text{ and } y_1 = \frac{x_0}{y_0} < \frac{\bar{x}}{\bar{y}} = \bar{y}.$$

Consequently,

$$x_{2n} < \bar{x}, \;\; x_{2n-1} > \bar{x}, \;\; y_{2n} > \bar{y}, \;\text{ and }\; y_{2n-1} < \bar{y}, \;\text{ for }\; n \geq 1.$$

From

$$x_{2n+1} = \frac{\alpha_1 + y_{2n}}{\alpha_1 + y_{2n-1}} \cdot x_{2n-1}$$

and

$$x_{2n+2} = \frac{\alpha_1 + y_{2n+1}}{\alpha_1 + y_{2n}} \cdot x_{2n}$$

we see that $\{x_{2n+1}\}$ increases above $\bar{x}$ and $\{x_{2n}\}$ decreases below $\bar{x}$ and so in view of the fact that system (9) does not have any prime period-two solutions, we see that

$$x_{2n+1} \uparrow \infty \;\text{ and }\; x_{2n} \downarrow 0.$$

Also, in view of Theorem 1.1, we see that the sequences $\{y_{2n+1}\}$ and $\{y_{2n}\}$ are eventually monotonic. In view of

$$y_{2n+1}y_{2n} = x_{2n} \downarrow 0$$

and

$$y_{2n+2}y_{2n+1} = x_{2n+1} \uparrow \infty$$

it follows that

$$y_{2n+1} \downarrow 0 \text{ and } y_{2n} \uparrow \infty.$$

To complete the proof observe that when the solution $\{x_n, y_n\}$ is bounded, it follows from Theorem 1.1 that the sequence $\{y_n\}$ converges to a finite limit, and so, the sequence $\{x_n\}$ also converges to a finite limit. □

Next, we go out of order and present the proof of the boundedness character of the special case,

$$(23,36): \qquad \left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 + y_n}{x_n}, \\ y_{n+1} &= \frac{\beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n}, \end{aligned}\right\}, \qquad n = 0, 1, \ldots . \tag{10}$$

We normalized the first equation by taking the parameter $\gamma_1 = 1$, but we allowed all the 4 parameters $\beta_2,\ \gamma_2,\ B_2,\ C_2$ in the second equation so that they can reveal the pattern of the boundedness character of solutions.

Clearly, when

$$C_2\beta_2 = B_2\gamma_2$$

the second component y_n of every solution $\{x_n, y_n\}$ of system (10) is constant, for $n \geq 1$, and the component x_n is periodic with period two. Therefore in this case every solution of the system is bounded and for $n \geq 1$, it is periodic with period 2. In the next lemma we establish the boundedness characters of solutions of system (10) when

$$C_2\beta_2 \neq B_2\gamma_2.$$

The equilibrium points of system (10) are the points $(\bar{x}, \bar{y})$ which satisfy

$$\bar{x} = \frac{\alpha_1 + \bar{y}}{\bar{x}}, \qquad \bar{y} = \frac{\beta_2\bar{x} + \gamma_2\bar{y}}{B_2\bar{x} + C_2\bar{y}}.$$

One can show that there is only one positive equilibrium point for system (1).

Lemma 4.2. *(a) Assume that*

$$C_2\beta_2 < B_2\gamma_2.$$

Then every solution of system (1) is bounded. Furthermore, every solution converges to a (not necessarily prime) period-two solution.

(b) Assume that

$$C_2\beta_2 > B_2\gamma_2.$$

Then system (10) has unbounded solutions. More specifically, for any initial *conditions* (x_0, y_0) *such that*

$$(x_0 - \bar{x})(y_0 - \bar{y}) < 0,$$

the solution $\{x_n, y_n\}$ *of system (10) is as follows:*

$$\lim_{n\to\infty} x_{2n} = 0, \quad \lim_{n\to\infty} x_{2n+1} = \infty, \quad \lim_{n\to\infty} y_{2n} = \beta_2/B_2, \quad \text{and} \quad \lim_{n\to\infty} y_{2n+1} = \gamma_2/C_2,$$

or

$$\lim_{n\to\infty} x_{2n} = \infty, \quad \lim_{n\to\infty} x_{2n+1} = 0, \quad \lim_{n\to\infty} y_{2n} = \gamma_2/C_2, \quad \text{and} \quad \lim_{n\to\infty} y_{2n+1} = \beta_2/B_2.$$

The proof of this result is under the assumption that all parameters in the system (1) are positive. Later on, this assumption can be partially released, as we will see that some parameters can be zeros.

Proof. Let $\{x_n, y_n\}$ be a positive solution of system (1). Then, clearly

$$\frac{\min\{\beta_2, \gamma_2\}}{\max\{B_2,\ C_2\}} < y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n} < \frac{\max\{\beta_2, \gamma_2\}}{\min\{B_2,\ C_2\}}, \quad n = 0, 1, \ldots$$

and so the component $\{y_n\}$ of the solution $\{x_n, y_n\}$ is bounded from above and from below by positive constants.

(a) Assume for the sake of contradiction that there exists a sequence of indices n_i such that

$$x_{n_i+1} \to \infty \quad \text{and} \quad x_{n_i+1} > x_j, \ \text{ for all } \ j < n_i + 1.$$

Then, clearly,

$$x_{n_i}, x_{n_i-2} \to 0 \ \text{ and } \ x_{n_i-1} \to \infty.$$

From this and in view of

$$y_{n_i} = \frac{\beta_2 x_{n_i-1} + \gamma_2 y_{n_i-1}}{B_2 x_{n_i-1} + C_2 y_{n_i-1}} \quad \text{and} \quad y_{n_i-1} = \frac{\beta_2 x_{n_i-2} + \gamma_2 y_{n_i-2}}{B_2 x_{n_i-2} + C_2 y_{n_i-2}}$$

it follows that

$$y_{n_i} \to \beta_2/B_2 \quad \text{and} \quad y_{n_i-1} \to \gamma_2/C_2.$$

Therefore eventually,

$$y_{n_i} < y_{n_i-1}.$$

Note that

$$x_{n_i+1} = \frac{\alpha_1 + y_{n_i}}{\alpha_1 + y_{n_i-1}} \cdot x_{n_i-1}$$

and so eventually

$$x_{n_i+1} < x_{n_i-1}$$

which is a contradiction. Finally, by using Theorem 1.1 one can see that the solutions of system converge to a (not necessarily prime) period-two solution. The proof of part (a) is complete.

(b) The proof is along the lines of the proof of Lemma 4.1 and will be omitted. □

One can slightly modify the proof of Lemma 4.1 to establish the boundedness characterization of systems: (23,15), (23,18), and (23,26), see Appendix II.

In the special case,

$$(23,27):\qquad \left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 + y_n}{x_n}, \\ y_{n+1} &= \frac{\beta_2 x_n + \gamma_2 y_n}{x_n}, \end{aligned}\right\}, \qquad \text{for } \; n = 0, 1, \dots, \tag{11}$$

we have the following result.

Lemma 4.3. *Every solution of system (11) converges to a finite limit.*

Proof. We first show that every solution of system (11) is bounded. Let $\{x_n, y_n\}$ be a positive solution of system (11). From the second equation of the system, we see that the sequence $\{y_n\}$ is bounded from below. From

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\alpha_1 + y_n}{\beta_2 x_n + \gamma_2 y_n}$$

it follows that the sequence $\{\frac{x_{n+1}}{y_{n+1}}\}$ is bounded and so

$$\frac{y_{n+1}}{x_{n+1}} = \frac{\beta_2 x_n + \gamma_2 y_n}{\alpha_1 + y_n}$$

is also bounded, and so clearly, the second component $\{y_n\}$ is bounded from above and from below by positive constants. From this it follows that the first component $\{x_n\}$ is also bounded.

Let $z_n = \frac{x_n}{y_n}$. Then system (11) can be transformed into the single equation:

$$z_{n+1} = \frac{\gamma_2 + (\alpha_1 + \beta_2) z_{n-1}}{(\beta_2 z_{n-1} + \gamma_2)(\beta_2 z_n + \gamma_2)}, \quad n = 0, 1, \dots .$$

One can easily see that the hypotheses of Theorem 1.1 are satisfied, and so, $\{z_{2n}\}$ and $\{z_{2n+1}\}$ are eventually monotonic subsequences. From

$$y_{n+1} = \beta_2 + \frac{\gamma_2}{z_n}, \quad \text{for } \; n \geq 0$$

we see that $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are eventually monotonic. The result is now a consequence of the fact that system (11) has no prime period-two solutions. □

5. Necessary and Sufficient Conditions for Boundedness when $\gamma_1 > 0$

A review of the results which we obtained in Sections 3 and 4 about the boundedness character of solutions of system (1) when

$$\gamma_1 > 0, \tag{12}$$

reveal the following pattern of boundedness for system (1):

Every solution of system (1) is bounded if and only if $C_2\beta_2 \leq B_2\gamma_2$.

More precisely, we established the following result:

Theorem 5.1. *Assume that (12) holds. Then the following statements are true for the solutions of system (1):*

(a) Assume that

$$C_2\beta_2 \leq B_2\gamma_2.$$

Then every solution of system (1) is bounded.

(b) Assume that

$$C_2\beta_2 > B_2\gamma_2.$$

Then for some initial conditions, system (1) has unbounded solutions.

References

[1] A.M. Amleh, E. Camouzis, G. Ladas, On second-order rational difference equations, Part 1, *J. Difference Equ. Appl.*, **13** (2007), 969-1004.

[2] A.M. Amleh, E. Camouzis, G. Ladas, On second-order rational difference equations, Part 2, *J. Difference Equ. Appl.*, **14** (2008), 215-228.

[3] A.M. Amleh, E. Camouzis, G. Ladas, On the dynamics of a rational difference equation, Part 1, *Int. J. Difference Equ.*, **3** (2008).

[4] A.M. Amleh, E. Camouzis, G. Ladas, On the dynamics of a rational difference equation, Part 2, *Int. J. Difference Equ.* (2008).

[5] E. Camouzis, Boundedness of solutions of a rational system of difference equations, In: *Proceedings of the 14-th ICDEA*, To Appear.

[6] E. Camouzis, M.R.S. Kulenović, G. Ladas, O. Merino, Rational systems in the plane, *J. Difference Equ. Appl.*, **15** (2009), 303-323.

[7] E. Camouzis, G. Ladas, *Dynamics of Third-Order Rational Difference Equations; With Open Problems and Conjectures*, Chapman and Hall/CRC Press (2008).

[8] E. Camouzis, G. Ladas, When does local stability imply global attractivity in rational equations?, *J. Difference. Equ. Appl.*, **12** (2006), 863-885.

[9] E. Camouzis, G. Ladas, Global results on rational systems in the plane, I, *J. Difference. Equ. Appl.* (2009).

[10] D. Clark, M.R.S. Kulenović, On a coupled system of rational difference equations, *Comput. Math. Appl.* **43** (2002), 849-867.

[11] D. Clark, M.R.S. Kulenović, J.F. Selgrade, Global asymptotic behavior of a two dimensional difference equation modelling competition, *Nonlinear Anal., TMA* **52** (2003), 1765-1776.

[12] C. A. Clark, M.R.S. Kulenović, J.F. Selgrade, On a system of rational difference equations, *J. Differ. Equations Appl.* **11** (2005), 565-580.

[13] J.M. Cushing, S. Levarge, N. Chitnis, S.M. Henson, Some discrete competition models and the competitive exclusion principle, *J. Differ. Eq. Appl.*, **10** (2004), 1139-1152.

[14] S. Elaydi, *An Introduction to Difference Equations*, Second Edition, Springer-Verlag, New York (1999).

[15] H.A. El-Metwally, E.A. Grove, G. Ladas, A global convergence result with applications to periodic solutions, *J. Math. Anal. Appl.*, **245** (2000), 161-170.

[16] H.A. El-Metwally, E.A. Grove, G. Ladas, H.D. Voulov, On the global attractivity and the periodic character of some difference equations, *J. Difference Equ. Appl.*, **7** (2001), 837-850.

[17] E.A. Grove, Y. Kostrov, G. Ladas, S.W. Schultz, Riccati difference equations with real period-2 coefficients, *Commun. Appl. Nonlinear Anal.*, **14** (2007), 33-56.

[18] E.A. Grove, G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman and Hall/CRC Press (2005).

[19] J.E. Franke, J.T. Hoag, G. Ladas, Global attractivity and convergence to a two-cycle in a difference equation, *J. Difference Equ. Appl.*, **5** (1999), 203-210.

[20] M. Hirsch, H.L. Smith, Monotone maps: A review, *J. Differ. Equ. Appl.*, **11** (2005), 379-398.

[21] W.G. Kelley, A.C. Peterson, *Difference Equations*, Academic Press, New York (1991).

[22] V.L. Kocic, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht (1993).

[23] M.R.S. Kulenović, G. Ladas, *Dynamics of Second Order Rational Difference Equations; with Open Problems and Conjectures*, Chapman and Hall/CRC Press (2001).

[24] M. R. S. Kulenović, M. Nurkanović, Asymptotic behavior of a competitive system of linear fractional difference equations, *Advances in Difference Equations*, **3** (2006), 1-13.

[25] E. Magnucka-Blandzi, J. Popenda, On the asymptotic behavior of a rational system of difference equations, *J. Difference Equ. Appl.*, **5**, No. 3 (1999), 271-286.

[26] H. Sedaghat, *Nonlinear Difference Equations, Theory and Applications to Social Science Models*, Kluwer Academic Publishers, Dordrecht (2003).

Appendix I

2.	$x_{n+1} = \frac{1}{x_n}$	8.	$x_{n+1} = \frac{\gamma_1 y_n}{x_n}$	23.	$x_{n+1} = \frac{\alpha_1+\gamma_1 y_n}{x_n}$

Table 1: The first equation of the system

5.	$y_{n+1} = \beta_2$	9.	$y_{n+1} = \gamma_2$	26.	$y_{n+1} = \frac{\beta_2 x_n+\gamma_2 y_n}{y_n}$
6.	$y_{n+1} = \frac{\gamma_2 y_n}{x_n}$	15.	$y_{n+1} = \frac{\gamma_2 y_n}{B_2 x_n+y_n}$	27.	$y_{n+1} = \frac{\beta_2 x_n+\gamma_2 y_n}{x_n}$
8.	$y_{n+1} = \frac{\beta_2 x_n}{y_n}$	18.	$y_{n+1} = \frac{\beta_2 x_n}{B_2 x_n+y_n}$	36.	$y_{n+1} = \frac{\beta_2 x_n+\gamma_2 y_n}{B_2 x_n+y_n}$

Table 2: The second equation of the system

Appendix II

(2,5)	$x_{n+1}=\frac{1}{x_n}$,	$y_{n+1}=\beta_2$	ESP2;	(B,B)
(2,6)	$x_{n+1}=\frac{1}{x_n}$,	$y_{n+1}=\frac{\gamma_2 y_n}{x_n}$	ESCP2 iff $\gamma_2\leq 1$;	(B,B) iff $\gamma_2\leq 1$ (B,U) iff $\gamma_2>1$
(2,8)	$x_{n+1}=\frac{1}{x_n}$,	$y_{n+1}=\frac{x_n}{y_n}$	ESP2 iff $x_0=1$;	(B,B) iff $x_0=1$ (B,U) iff $x_0\neq 1$
(2,9)	$x_{n+1}=\frac{1}{x_n}$,	$y_{n+1}=\gamma_2$	ESP2;	(B,B)
(2,15)	$x_{n+1}=\frac{1}{x_n}$,	$y_{n+1}=\frac{\gamma_2 y_n}{x_n+y_n}$	ESCP2;	(B,B)
(2,18)	$x_{n+1}=\frac{1}{x_n}$,	$y_{n+1}=\frac{\beta_2 x_n}{x_n+y_n}$	ESCP2;	(B,B)
(2,26)	$x_{n+1}=\frac{1}{x_n}$,	$y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{y_n}$	ESCP2	(B,B)
(2,27)	$x_{n+1}=\frac{1}{x_n}$,	$y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{x_n}$	ESCP2 iff $\gamma_2<1$	(B,B) iff $\gamma_2<1$ (B,U) iff $\gamma_2\geq 1$
(2,36)	$x_{n+1}=\frac{1}{x_n}$,	$y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{x_n+y_n}$	ESCP2;	(B,B)
(8,5)	$x_{n+1}=\frac{y_n}{x_n}$,	$y_{n+1}=\beta_2$	ESP2;	(B,B)
(8,6)	$x_{n+1}=\frac{y_n}{x_n}$,	$y_{n+1}=\frac{y_n}{x_n}$	ESC;	(B,B)
(8,8)	$x_{n+1}=\frac{y_n}{x_n}$,	$y_{n+1}=\frac{x_n}{y_n}$	ESC iff $x_0=y_0$	(U,U) iff$x_0\neq y_0$ (B,B) iff$x_0=y_0$
(8,9)	$x_{n+1}=\frac{y_n}{x_n}$,	$y_{n+1}=\gamma_2$	ESP2;	(B,B)
(8,15)	$x_{n+1}=\frac{y_n}{x_n}$,	$y_{n+1}=\frac{\gamma_2 y_n}{x_n+y_n}$	ESC;	(B ,B)
(8,18)	$x_{n+1}=\frac{y_n}{x_n}$,	$y_{n+1}=\frac{\beta_2 x_n}{x_n+y_n}$	$z_{n+1}=\frac{1+z_n}{\beta_2 z_n^2}$	(U, B)
(8,26)	$x_{n+1}=\frac{y_n}{x_n}$,	$y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{y_n}$	$z_{n+1}=\frac{1}{z_n(\beta_2 z_n+\gamma_2)}$	(U,U)
(8,27)	$x_{n+1}=\frac{y_n}{x_n}$,	$y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{x_n}$	ESC;	(B,B)
(8,36)	$x_{n+1}=\frac{y_n}{x_n}$,	$y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{x_n+y_n}$	ESC iff $\mathcal{D}<0$ ESP2 iff$\mathcal{D}=0$	(B,B) iff $\mathcal{D}<0$ (U,B) iff $\mathcal{D}>0$
(23,5)	$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{x_n}$,	$y_{n+1}=\beta_2$	ESP2;	(B,B)
(23,6)	$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{x_n}$,	$y_{n+1}=\frac{y_n}{x_n}$	ESC;	(B,B)
(23,8)	$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{x_n}$,	$y_{n+1}=\frac{x_n}{y_n}$	EBSC; $z_{n+1}=\frac{\alpha_1+\gamma_1 z_{n-1}}{z_{n-1}z_n^2}$	(U,U)
(23,9)	$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{x_n}$,	$y_{n+1}=\gamma_2$	ESP2;	(B,B)
(23,15)	$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{x_n}$,	$y_{n+1}=\frac{\gamma_2 y_n}{x_n+y_n}$	ESCP2; $z_{n+1}=$ $\frac{((\gamma_1\gamma_2+\alpha_1)+\alpha_1 z_{n-1})(1+z_n)}{\gamma_2^2 z_{n-1}z_n}$	(B,B)
(23,18)	$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{x_n}$,	$y_{n+1}=\frac{\beta_2 x_n}{x_n+y_n}$	$z_{n+1}=$ $\frac{(\alpha_1+(\alpha_1+\gamma_1\beta_2)z_{n-1})(1+z_n)}{\beta_2^2 z_{n-1}z_n^2}$	(U,B)

(23,26) $x_{n+1} = \frac{\alpha_1+\gamma_1 y_n}{x_n}$, $y_{n+1} = \frac{\beta_2 x_n+\gamma_2 y_n}{y_n}$	$z_{n+1} = \frac{(\gamma_1\gamma_2+\alpha_1)+\gamma_1\beta_2 z_{n-1}}{(\beta_2 z_{n-1}+\gamma_2)z_n(\beta_2 z_n+\gamma_2)}$	(U, U)
(23,27) $x_{n+1} = \frac{\alpha_1+\gamma_1 y_n}{x_n}$, $y_{n+1} = \frac{\beta_2 x_n+\gamma_2 y_n}{x_n}$	ESC; $z_{n+1} = \frac{\gamma_1\gamma_2+(\alpha_1+\gamma_1\beta_2)z_{n-1}}{(\beta_2 z_{n-1}+\gamma_2)(\beta_2 z_n+\gamma_2)}$	(B, B)
(23,36) $x_{n+1} = \frac{\alpha_1+\gamma_1 y_n}{x_n}$, $y_{n+1} = \frac{\beta_2 x_n+\gamma_2 y_n}{x_n+y_n}$	ESC if $D < 0$; $z_{n+1} = \frac{((\gamma_1\gamma_2+\alpha_1)+(\alpha_1+\gamma_1\beta_2)z_{n-1}(1+z_n))}{(\beta_2 z_{n-1}+\gamma_2)z_n(\beta_2 z_n+\gamma_2)}$	(U,B) iff $\mathcal{D} > 0$ (B,B) iff $\mathcal{D} < 0$

The List of 27 special cases of the system

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{x_n}, \qquad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n},$$

$$\mathcal{D} = C_2\beta_2 - B_2\gamma_2$$