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NOVEL FINITE DIFFERENCE SCHEMES FOR THIRD ORDER BOUNDARY VALUE PROBLEMS

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Abstract: Multiple finite difference methods (MFDMs) are obtained from a continuous k-step linear multistep method (LMM) and applied to solve third order boundary value problems (BVPs). Convergence of the MFDMs is established through consistency and zero-stability by expressing them as a block method. The intervals of absolute stability for the methods are calculated using the boundary locus technique. Numerical experiments are performed to show the efficiency of the methods.

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1. Introduction

Third order two-point BVPs frequently occur in physical sciences and engineering. For instance, the draining and coating flow problems can be formulated as a third order BVP as described in Tuck and Schwartz [27]. The popular Blasius equation which originates from the theory of the laminar boundary layer and sandwich beam problems are modeled as third order BVPs, see [22], [24], and [26]. Several important applications in fluid dynamics are reported as third BVPs as discussed in Valarmathi and Ramanujam [29]. It is common knowl-

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edge that several differential equations cannot be solved analytically, hence the development of new numerical techniques continue to be crucial. Thus, in this paper, we propose MFDMs for the general third-order BVP

$$Dy \equiv y''' = f(x, y, y', y'') \tag{1}$$

subject to any of the following boundary conditions:

$$y(a) = y_0,\ y'(a) = \delta_0,\ y(b) = y_N\,,$$
$$y(a) = y_0,\ y'(a) = \delta_0,\ y'(b) = y_{N_1}\,,$$
$$y(a) = y_0,\ y''(a) = \gamma_0,\ y''(b) = y_{N_2}\,,$$

where $a, b, \delta_0, \gamma_0, y_0, y_N, y_{N_1}, y_{N_2}$ are real constants. We note that f satisfies a Lipschitz condition and Keller [15] has given the theorem and the proof of the general conditions which ensure that the solution to (1) will exist and be unique. We emphasize that the MFDMs can be extended without any difficulty to solve other problems associated with mixed boundary conditions such as $\kappa_1 y''(a)+\kappa_2 y'(a) = \delta_{01},\ \kappa_3 y''(b)+\kappa_4 y'(b) = y_{N_{01}},\ y(b) = y_N$ where $|\kappa_1|+|\kappa_2| \neq 0,\ \ |\kappa_3| + |\kappa_4| \neq 0$, and $\kappa_1, \delta_{01}, N_{01}, \kappa_2, \kappa_3, \kappa_4$ are real constants. Moreover, the MFDMs can be used to solve problems associated with free boundary conditions as in the case of the Blasius problem. It is worthwhile to note that the MFDMs are not restricted to the scalar form (1) since they can be extended to solve a system of such equations by obvious notational modifications.

Third order BVPs are conventionally solved by the standard finite difference methods (SFDMs), which have been considered by authors such as Collatz [4], Fox [7], and Henrici [9]. More accurate methods based on the use of splines, non-polynomial splines, and high order finite difference methods suggested in the literature for solving third order BVPs are due to Khan and Aziz [16], Usmani and Sakai [28], Al-Said and Noor [23], Siraj-ul-Islam and Tirmizi [25] as well as Salama and Mansour [22]. The method suggested in this paper is based on a continuous LMM. LMMs are very popular for solving first order initial value problems (IVPs) and are conventionally used to solve higher order IVPs by first reducing it to an equivalent system of first-order ordinary differential equations (ODEs). This approach is extensively discussed by authors such as Lambert [17], [18], Brugnano and Trigiante [3], Onumanyi et al [21], [20], Fatunla [6], and Jennings [14]. LMMs of the Adams-Moulton type for solving BVPs are due to Onumanyi et al [20], and Brugnano and Trigiante [3]. However, these methods are applicable by first reducing the higher order ODE to an equivalent system of first-order ODEs, which involves more human effort and computer time as discussed in Awoyemi [2]. We note that LMMs based on the Numerov's type method have been considered by Yusuph and Onumanyi [30], Lambert

[18], and Henrici [9] for solving directly the special case $y'' = f(x, y)$ subject to Dirichlet boundary conditions.

Recently, Jator and Li [11], Jator [10], [12] proposed LMMs for the direct solution of the general second and third order IVPs, which were shown to be zero stable and implemented without the need for either predictors or starting values from other methods. Jator [13] used the LMMs developed for IVPs and additional methods obtained from the same continuous k-step LMM to solve third order BVPs with Dirichlet and Neumann boundary conditions. Therefore, we are motivated by the results in [13] to develop more accurate MFDMs with large intervals of absolute stability for (1) subject to Dirichlet, Neumann, and Robin boundary conditions.

In particular, a continuous k-step LMM is derived and used to generate multiple finite difference methods, which are assembled and applied as a single matrix of finite difference equations to (1) over sub-intervals which do not overlap as discussed in [30]. It is worth noting that the simultaneous application of these MFDMs are more accurate than SFDMs which are generally applied as single formulas over overlapping intervals as in Lambert [17] and Jennings [14]. In addition, higher order SFDMs are more tedious to derive and implement since in some cases higher order derivatives are not always easy to obtain. Thus, the method presented in this paper is more robust than the SFDMs. Convergence of the MFDMs is established through consistency and zero-stability by expressing them as a block method. The intervals of absolute stability for the methods are calculated using the boundary locus technique. We emphasize that the main method is derived through interpolation and collocation, see Lie and Norsett [19], Atkinson [1], Onumanyi et al [20]. The approach facilitates the link between the the finite difference methods and the k-step multistep collocation procedure, which are two important global methods which have been used with piecewise continuous approximate solution of ordinary differential equations (ODEs) Gladwell and Sayers [8].

The paper is organized as follows. In Section 2, we derive a continuous approximation $Y(x)$ for the exact solution $y(x)$. Section 3 is devoted to the specification of the methods and how the MFDMs are obtained. The analysis and implementation of the MFDMs are discussed in Section 4. Numerical examples are given in Section 5 to show the efficiency of the MFDMs. Finally, the conclusion of the paper is discussed in Section 6.

2. Derivation of the Method

In this section, we approximate the exact solution $y(x)$ by seeking the continuous method $Y(x)$ of the form

$$Y(x) = \sum_{j=0}^{r+s-1} \lambda_j \Upsilon_j(x)\,, \tag{2}$$

where $x \in [a,b]$, λ_j's are unknown coefficients and $\Upsilon_j(x)$'s are polynomial basis functions of degree $r+s-1$. The number of interpolation points r and the number of distinct collocation points s are chosen to satisfy $3 \leq r \leq k$, and $0 < s \leq k+1$ respectively. The positive integer $k \geq 3$ denotes the step number of the method. In what follows, we adopt a procedure for the determination of λ_j's which are used to construct a k-step multistep collocation method by imposing the following conditions.

$$Y\left(x_{n+j}\right) = y_{n+j}, \qquad j = 0,1,2,\ldots,r-1\,, \tag{3}$$

$$DY\left(x_{n+j}\right) = f_{n+j}, \qquad j = 0,1,2,\ldots,s-1\,. \tag{4}$$

We note that y_{n+j} is the numerical approximation to the analytical solution $y(x_{n+j})$, $f_{n+j} = f(x_{n+j}, y_{n+j})$, and n is a grid index.

Equations (3) and (4) lead to a system of $(r+s)$ equations, which is solved to obtain the λ_j's. We proceed by considering the following notations.

We define the interpolation/collocation matrix $\bigvee$ of dimension $(r+s)\times(r+s)$ as

$$\bigvee = \begin{pmatrix} P_0(x_n) & \cdots & P_{r+s-1}(x_n) \\ P_0(x_{n+1}) & \cdots & P_{r+s-1}(x_{n+1}) \\ \vdots & & \vdots \\ P_0(x_{n+r-1}) & \cdots & P_{r+s-1}(x_{n+r-1}) \\ DP_0(x_n) & \cdots & DP_{r+s-1}(x_n) \\ DP_0(x_{n+1}) & \cdots & DP_{r+s-1}(x_{n+1}) \\ \vdots & & \vdots \\ DP_0(x_{n+s-1}) & \cdots & DP_{r+s-1}(x_{n+s-1}) \end{pmatrix},$$

and consider further notations by defining the following vectors:

$\bigwedge = (y_n, y_{n+1}, \ldots, y_{n+r-1}, f_n, f_{n+1}, \ldots, f_{n+s-1})^T$,

$\Upsilon(x) = (P_0(x), P_1(x), \ldots, P_{r+s-1}(x))^T$,

$\bigsqcup = (\lambda_0, \lambda_1, \ldots, \lambda_{r+s-1})^T$,

where T denotes the transpose of the vectors. It worth noting that $\Upsilon(x)$ represents a vector of arbitrary basis functions.

The collocation points are selected from the extended set Ψ, where

$$\Psi = \{x_n, \ldots, x_{n+k}\} \bigcup \{x_{n+k-1}, x_{n+k}\}.$$

Theorem 2.1. *Let $Y(x)$ satisfy conditions (3) and (4), then, the continuous k-step LMM is constructed from the following equation:*

$$Y(x) = \bigwedge^{T} \left(\bigvee^{-1}\right)^{T} \Upsilon(x).$$

Proof. We begin the proof by first writing (2) in vector form as follows:

$$Y(x) = \bigsqcup{}^{T} \Upsilon(x). \tag{5}$$

We also write the system obtained from (3) and (4) in matrix form as follows:

$$\bigvee \bigsqcup = \bigwedge. \tag{6}$$

We assume that $\bigvee$ is non-singular and hence invertible, it follows from (6) that

$$\bigsqcup = \bigvee{}^{-1} \bigwedge. \tag{7}$$

It follows from (5) and (7) that

$$Y(x) = \bigwedge^{T} \left(\bigvee^{-1}\right)^{T} \Upsilon(x).$$

The proof is complete. ∎

The k-step LMM is obtained from Theorem 2.1 after some manipulation and expressed in the form

$$Y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^3 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j}, \tag{8}$$

where the $\alpha_j(x)$'s and $\beta_j(x)$'s are the continuous coefficients. The continuous k-step LMM (8) is used to generate MFDMs, which are applied as simultaneous numerical integrators to provide the discrete solution to (1). In this light, we seek a solution on the mesh

$$\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots < x_N = b,$$

where π_N is a partition of $[a, b]$ and $h = (b - a)/N$ is the constant step-size of the partition of π_N.

3. Specification of the Methods

In this section, we use (8) and the formulas for the derivatives which are expressed as

$$Y'(x) = \frac{1}{h}\left(\sum_{j=0}^{r-1} \alpha'_j(x) y_{n+j} + h^3 \sum_{j=0}^{s-1} \beta'_j(x) f_{n+j}\right), \tag{9}$$

$$Y''(x) = \frac{1}{h^2}\left(\sum_{j=0}^{r-1} \alpha''_j(x) y_{n+j} + h^3 \sum_{j=0}^{s-1} \beta''_j(x) f_{n+j}\right), \tag{10}$$

which provide additional equations and derivatives obtained by imposing that

$$Y'(x) = \delta(x), \qquad Y''(x) = \gamma(x)\,, \tag{11}$$

$$Y'(a) = \delta_0, \qquad Y''(a) = \gamma_0\,, \tag{12}$$

sto generate MFDMs for $k = 4$

In particular, we use (8) to obtain a continuous k-step LMM by specifying r, s, k, and $\Upsilon_j(x)$. We emphasize that the main method is obtained by evaluating (8) at $x = x_{n+k}$. We also express $\alpha_j(x)$ and $\beta_j(x)$ as functions of t for convenience, where $t = (x - x_{n+k-1})/h$. The coefficients $\alpha'_j(x)$, $\alpha''_j(x)$, $\beta'_j(x)$, and $\beta''_j(x)$ are easily obtained from the first and second derivatives of $\alpha_j(x)$ and $\beta_j(x)$. We discuss details of a specific method next.

Case $k = 4$. We use (8) to obtained a continuous 4-step method with the following specifications: $r = 3$, $s = 5$, $k = 4$, $\Upsilon_i(x) = x^i$, $i = 0, 1, \cdots, 7$. We also express $\alpha_j(x)$ and $\beta_j(x)$ as functions of t where $t = (x - x_{n+3})/h$ in what follows.

$\alpha_0(t) = \frac{1}{2}(2 + 3t + t^2)$,
$\alpha_1(t) = -3 - 4t - t^2$,
$\alpha_2(t) = \frac{1}{2}(6 + 5t + t^2)$,
$\beta_0(t) = \frac{1}{10080}(42 + 89t + 70t^2 - 35t^4 - 7t^5 + 7t^6 + 2t^7)$,
$\beta_1(t) = \frac{1}{5040}(2436 + 3602t + 1092t^2 + 105t^4 + 14t^5 - 21t^6 - 4t^7)$,
$\beta_2(t) = \frac{1}{1680}(882 + 1755t + 973t^2 - 105t^4 + 7t^5 + 14t^6 + 2t^7)$,
$\beta_3(t) = \frac{1}{5040}(-84 + 326t + 1036t^2 + 840t^3 + 175t^4 - 70t^5 - 35t^6 - 4t^7)$,
$\beta_4(t) = \frac{1}{10080}(42 + 5t - 84t^2 + 105t^4 + 77t^5 + 21t^6 + 2t^7)$.

First, we evaluate (8) at $x = \{x_{n+4}, x_{n+3}\}$ to obtain the main method and one additional method as follows:

$$y_{n+4}-6y_{n+2}+8y_{n+1}-3y_n=\frac{h^3}{60}(f_n+86f_{n+1}+126f_{n+2}+26f_{n+3}+f_{n+4})\,, \quad (13)$$

$$y_{n+3}-3y_{n+2}+3y_{n+1}-y_n=\frac{h^3}{240}(f_n+116f_{n+1}+126f_{n+2}-4f_{n+3}+f_{n+4})\,. \quad (14)$$

Since $k=4$, we need two additional methods which can be combined with (13) and (14) to simultaneously solve third order BVPs. Hence, we invoke the following remark which emphasizes how two additional methods are obtained from (11).

Remark 3.1. The functions $\delta(x)$ and $\gamma(x)$ are piecewise continuous on the sub-intervals $[x_0,x_k],\ldots,[x_{N-k},x_N]$ of $[a,b]$ with matching points at $x_{n+k}, n=0,k,2k,\ldots,N-2k$.

From Remark 3.1, for $k=4$, we impose that $\delta(x)$ and $\gamma(x)$ are continuous at $x=x_{n+4}$. Hence, the following two additional methods

$$\begin{aligned} y_{n+6}-4y_{n+5}+3y_{n+4}+7y_{n+2}-12y_{n+1}+5y_n=\frac{h^3}{2520}(-55f_n-6122f_{n+1}\\ -10242f_{n+2}-5030f_{n+3}-84f_{n+4}+1586f_{n+5}-342f_{n+6}+158f_{n+7}\\ -29f_{n+8})\,, \end{aligned} \quad (15)$$

$$\begin{aligned} y_{n+6}-2y_{n+5}+y_{n+4}-y_{n+2}+2y_{n+1}-y_n=\frac{h^3}{720}(-9f_n+418f_{n+1}+570f_{n+2}\\ +942f_{n+3}+472f_{n+4}+606f_{n+5}-186f_{n+6}+82f_{n+7}-15f_{n+8}) \end{aligned} \quad (16)$$

are obtained from (11) with continuity equations imposed at $x=x_{n+4}$ as follows:

$$\lim_{x\to x_{n+4}^-}\delta(x)=\lim_{x\to x_{n+4}^+}\delta(x), \qquad \lim_{x\to x_{n+4}^-}\gamma(x)=\lim_{x\to x_{n+4}^+}\gamma(x)\,,$$

where

$$\delta(x)=\begin{cases} \frac{1}{h}(\frac{5}{2}y_n-6y_{n+1}+\frac{7}{2}y_{n+2}+\frac{h^3}{5040}(55f_n+6122f_{n+1}+10242f_{n+2}\\ \quad +5030f_{n+3}+391f_{n+4})), \quad x_n\le x\le x_{n+4},\\ \frac{1}{h}(-\frac{3}{2}y_{n+4}+2\ y_{n+5}-\frac{1}{2}y_{n+6}+\frac{h^3}{5040}(307f_{n+4}\\ \quad +1586f_{n+5}-342f_{n+6}+158f_{n+7}-29f_{n+8})),\\ \quad x_{n+4}\le x\le x_{n+8}, \end{cases} \quad (17)$$

$$\gamma(x) = \begin{cases} \frac{1}{h^2}(y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{720}(-9f_n + 418f_{n+1} \\ +570f_{n+2} + 942f_{n+3} + 239f_{n+4}), \quad x_n \le x \le x_{n+4}\,, \\ \frac{1}{h^2}(y_{n+4} - 2y_{n+5} + y_{n+6} + \frac{h^3}{720}(-233f_{n+4} - 606f_{n+5} \\ +186f_{n+6} - 82f_{n+7} + 15f_{n+8})), \quad x_{n+4} \le x \le x_{n+8}\,. \end{cases} \tag{18}$$

It is worth noting that the derivatives are provided by $\delta(x_{n+\tau}) = \delta_{n+\tau}$, $\gamma(x_{n+\tau}) = \gamma_{n+\tau}$, $\tau = 0, \dots, 4$ as follows:

$$h\delta_n = -\frac{1}{2}y_{n+2} + 2y_{n+1} - \frac{3}{2}y_n + \frac{h^3}{5040}(307f_n + 1586f_{n+1} - 342f_{n+2} + 158f_{n+3} - 29f_{n+4})\,,$$

$$h\delta_{n+1} = \frac{1}{2}y_{n+2} - \frac{1}{2}y_n + \frac{h^3}{10080}(-79f_n - 1532f_{n+1} - 54f_{n+2} - 20f_{n+3} + 5f_{n+4})\,,$$

$$h\delta_{n+2} = \frac{3}{2}y_{n+2} - 2y_{n+1} + \frac{1}{2}y_n + \frac{h^3}{5040}(13f_n + 1166f_{n+1} + 582f_{n+2} - 94f_{n+3} + 13f_{n+4})\,,$$

$$h\delta_{n+3} = \frac{5}{2}y_{n+2} - 4y_{n+1} + \frac{3}{2}y_n + \frac{h^3}{10080}(89f_n + 7204f_{n+1} + 10530f_{n+2} + 652f_{n+3} + 5f_{n+4})\,,$$

$$h\delta_{n+4} = \frac{7}{2}y_{n+2} - 6y_{n+1} + \frac{5}{2}y_n + \frac{h^3}{5040}(55f_n + 6122f_{n+1} + 10242f_{n+2} + 5030f_{n+3} + 391f_{n+4})\,,$$

$$h^2\gamma_n = y_{n+2} - 2y_{n+1} + y_n + \frac{h^3}{720}(-233f_n - 606f_{n+1} + 186f_{n+2} - 82f_{n+3} + 15f_{n+4})\,,$$

$$h^2\gamma_{n+1} = y_{n+2} - 2y_{n+1} + y_n + \frac{h^3}{360}(9f_n + 20f_{n+1} - 39f_{n+2} + 12f_{n+3} - 2f_{n+4})\,,$$

$$h^2\gamma_{n+2} = y_{n+2} - 2y_{n+1} + y_n - \frac{h^3}{720}(-f_n + 386f_{n+1} + 378f_{n+2} - 50f_{n+3} + 7f_{n+4})\,,$$

$$h^2\gamma_{n+3} = y_{n+2} - 2y_{n+1} + y_n + \frac{h^3}{360}(5f_n + 156f_{n+1} + 417f_{n+2} + 148f_{n+3} - 6f_{n+4})\,,$$

$$h^2\gamma_{n+4} = y_{n+2} - 2y_{n+1} + y_n + \frac{h^3}{720}(-9f_n + 418f_{n+1} + 570f_{n+2} + 942f_{n+3} + 239f_{n+4})\,.$$

4. Analysis and Implementation of the Methods

The methods (13)-(16) are specified members of the conventional LMM which can be represented as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j} \,, \tag{19}$$

or compactly written in the form

$$\rho(E) y_n = h^3 \sigma(E) f_n \,, \tag{20}$$

where $\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j$ and $\sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j$ are the characteristic polynomials, $\zeta \,\epsilon\, \mathbb{C}$, and $E^j y_n = y_{n+j}$ is a shift operator.

Following Fatunla [5] and Lambert [17] we define the local truncation error associated with (19) to be the linear difference operator

$$L[y(x); h] = \sum_{j=0}^{k} \{\alpha_j y(x + jh) - h^3 \beta_j y'''(x + jh)\} \,. \tag{21}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (21) as a Taylor series about the point x to obtain the expression

$$L[y(x); h] = C_0 y(x) + C_1 h y' + \ldots, + C_q h^q y^q(x) + \ldots, \tag{22}$$

where the constant coefficients C_q , $q = 0, 1, \ldots$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j \,,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j \,,$$

$$C_2 = \frac{1}{2!} \sum_{j=1}^{k} j^2 \alpha_j \,,$$

$$\vdots$$

$$C_q = \frac{1}{q!} \left(\sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2) \sum_{j=1}^{k} j^{q-3} \beta_j \right) .$$

In the sense of Henrici [9], we say that the method (19) has order p if

$$C_0 = C_1 = \ldots = C_p = C_{p+1} = C_{p+2} = 0, \quad C_{p+3} \neq 0;$$

therefore, C_{p+3} is the error constant and $C_{p+3} h^{p+3} y^{(p+3)}(x_n)$ the principal local truncation error at the point x_n. The orders and error constants for the methods (13) - (16) were calculated using the coefficients $C_0, C_1, \ldots, C_p, C_{p+2}$ and C_{p+3} which are stated above. The details of the results which are displayed in Table 1 show that the methods are highly accurate, since they have high orders and small error constants.

In what follows, we discuss the regions of absolute stability via the boundary locus method by considering the test equation $Dy = \lambda y, \lambda, y \epsilon \mathbb{R}$. On applying (19) to the test equation yields the difference equation $(\rho(E) - \hbar\sigma(E))y_n = 0$, $\hbar = h^3\lambda$, $\lambda < 0$, whose solution can be generated by the roots of the stability polynomial $\Phi(\zeta, \hbar) = \rho(\zeta) - \hbar\sigma(\zeta)$. The roots of the stability polynomial are generally complex numbers, hence we assume the $\hbar$ is also a complex number. We define a region of stability Ω with boundary $\delta\Omega$ to be a region of the complex $\hbar$-plane such that the roots of $\Phi(\zeta, \hbar) = 0$ lie within the unit circle whenever $\hbar$ lies in the interior of the region (see [18]). Since the roots of $\Phi(\zeta, \hbar) = 0$ are continuous functions of $\hbar$, $\hbar$ will lie on $\delta\Omega$ when one of the roots of $\Phi(\zeta, \hbar) = 0$ lies on the boundary of the unit circle. Thus, $\Phi(e^{i\theta}, \hbar) = \rho(e^{i\theta}) - \hbar\sigma(e^{i\theta}) = 0$. Hence, the locus of $\delta\Omega$ is expressed as

$$\hbar(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})},$$

which after some manipulation can be written in the form $\hbar(\theta) = X_1(\theta) + iX_2(\theta)$. Our calculations show that methods (13) to (16) yield the vectors $X_1(0) = (0, 0, 0, 0)^T$, $X_1(\pi) = (-60, -120, -60, 0)^T$, $X_2(0) = (0, 0, 0, 0)^T$, $X_2(\pi) = (0, 0, 0, 0)^T$. We note that for real $\hbar$, the end-points of the intervals of absolute stability are given by the points at which $\delta\Omega$ cuts the real axis. The intervals of absolute stability for the methods (13)-(16) are summarized in Table 1.

Remark 4.1. Although the method (16) has an empty interval of absolute stability it is successfully and simultaneously used together with the methods (13)-(15) as numerical integrators for BVPs. We also note that the method (16) is highly accurate since it has order $p = 6$ and error constant $C_{p+3} = C_9 = -799/15120$. However, the method (16) is not recommended to be used singly in the conventional fashion, since starting values and predictors will be required, hence making its implementation very costly. In this paper, starting values and predictors are avoided, since our method is implemented efficiently by combining the MFDMs ((13)-(16)) as simultaneous numerical integrators for third order BVPs. The small absolute errors produced by the MFDMs (see Tables 2-4) validate the fact that the methods are highly accurate.

Method	Order p	Error constant C_{p+3}	Interval of Absolute Stability
(13)	5	$-1/240$	$(-60, 0)$
(14)	5	$-1/240$	$(-120, 0)$
(15)	5	$53/2520$	$(-60, 0)$
(16)	6	$-799/15120$	$(0, 0)$

Table 1: Orders, error constants, and intervals of absolute stability for MFDMs

In order to analyze the methods for zero-stability, we normalize the MFDMs and write them as a block method from which we obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(RA^0 - A^1) = R(R-1)^3\,, \tag{23}$$

where A^0 is an identity matrix of dimension k, A^1 is a matrix of dimension k and is given by

$$A^1 = \begin{pmatrix} 0 & 1 & -3 & 3 \\ 0 & 3 & -8 & 6 \\ 0 & 6 & -15 & 10 \\ 0 & 10 & -24 & 15 \end{pmatrix}.$$

Following Fatunla [5], the MFDMs are zero-stable, since from (23), $\rho(R) = 0$ satisfy $|R_j| \le 1$, $j = 1, \ldots, k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 3. The MFDMs are consistent since each has order $p > 1$. According to Henrici [9], we can safely assert the convergence of the MFDMs.

Our method is implemented efficiently by combining the MFDMs as simultaneous numerical integrators for third order BVPs. In particular, for linear problems, we can solve (1) directly from the start with Gaussian elimination using partial pivoting, and for nonlinear problems, we can use a modified Newton-Raphson method. In each case, the MFDMs are combined to give a single matrix of finite difference equations which simultaneously provides the values of the solution and the first and second derivatives generated by the sequences $\{y_n\}, \{y'_n\}, \{y''_n\}, n = 0, \ldots, N$, where the single block matrix equation is solved while adjusting for boundary conditions.

h	Our Method EMAX	Salama and Mansour [22] EMAX	Khan and Aziz [16] EMAX
1/8	1.20×10^{-7}	1.35×10^{-6}	1.84×10^{-6}
1/16	1.72×10^{-9}	9.61×10^{-8}	1.04×10^{-7}
1/32	2.68×10^{-11}	6.17×10^{-9}	6.32×10^{-9}
1/40	6.96×10^{-12}		
1/48	2.35×10^{-12}		
1/56	9.16×10^{-13}		
1/64	4.90×10^{-13}	3.88×10^{-10}	

Table 2: Maximum absolute error (EMAX) for Example 5.1

5. Numerical Examples

In this section, we have tested the performance of our method on four problems. For each example we find the maximum absolute error of the approximate solution in π_N for different values of N, where N is chosen to be divisible by k. All computations were carried out using our written *Mathematica* code in *Mathematica 6.0*.

Example 5.1. Our first test problem is the given special two-point special third order BVP.

$$y''' = xy + (x^3 - 2x^2 - 5x - 3)e^x, \quad y(0) = y(1) = 0, \; y'(0) = 1, \quad 0 \le x \le 1 .$$

$$\text{Exact}: \quad y(x) = x(1-x)e^x .$$

The maximum absolute errors are expressed as EMAX $= \text{Max}\,|y(x_i) - y_i|$, $i = 1, \ldots, N$, where $y(x_i)$ is the exact solution computed at the grid point and y_i is an approximation to the exact solution using MFDMs. For this example, our method is more accurate than those given in [22], [16]. The details of the numerical results are reported in Table 2.

Example 5.2. The second test problem is a general third order BVP which was also solved in [22].

$$y''' + 2y'' - 4y' + y = \frac{1}{4}(8x - x^2) + \frac{9e^{-2x} + 2x - 9}{4(1 - e^{-2})},$$
$$y(0) = y'(0) = 1, \quad y'(1) = 1, \quad 0 \le x \le 1 ,$$

$$\text{Exact}: \; y(x) = c_1 e^x + c_2 e^{\frac{-x}{2}(3-\sqrt{13})} + c_3 e^{\frac{-x}{2}(3+\sqrt{13})} + \frac{1}{4}(4 - x^2) + \frac{e^{-2x} + 2x - 1}{4(1 - e^{-2})} .$$

h	Our Method EMAX	Salama and Mansour [22] EMAX	SFDM EMAX
1/8	3.96×10^{-6}	2.69×10^{-7}	2.41×10^{-2}
1/16	8.16×10^{-8}	1.85×10^{-8}	4.56×10^{-3}
1/32	1.36×10^{-9}	1.17×10^{-9}	1.01×10^{-3}
1/40	3.59×10^{-10}		
1/48	1.21×10^{-10}		
1/56	4.86×10^{-11}		
1/64	2.14×10^{-11}	7.37×10^{-11}	2.39×10^{-4}

Table 3: Maximum absolute error, EMAX $=$ Max $|y(x_i) - y_i|$, $i = 1, \ldots, N$, $y(x_i)$ is the exact solution computed at the grid point and y_i is an approximation to the exact solution using MFDMs for Example 5.2

The numerical results of this problem were compared with the SFDM and the method developed by Salama and Mansour [22]. It is obvious from Table 3 that our method performs better than the standard difference method, however the method in [22] performs better than our method due to the fact the the BVP is subject to a Neumann boundary condition at b. Recall from the first example that our method performed better than the method in [22] when the BVP was subject to a Dirichlet boundary condition at b. However, our method performs better with increased number of steps, for instance, our method gives EMAX $= 2.14 \times 10^{-11}$ for $h = 1/64$, while the method in [22] gives EMAX $= 7.37 \times 10^{-11}$.

Example 5.3. Our third test problem is intended to demonstrate that our method can also be used to solve third order BVPs with mixed boundary conditions. Thus, we consider the following BVP.

$$\begin{aligned} &y''' + \frac{1}{x}y'' - \frac{1}{x^2}y' = \frac{1}{x}, \\ &y''(1) + 0.3y'(1) = 0, \qquad y''(2) + 0.15y'(2) = 0, \\ &y(2) = 0, \quad 1 \leq x \leq 2\,. \end{aligned}$$

$$\text{Exact}: \; y(x) = C_1 + C_2 \ln x + C_3 x^2 - \frac{1}{4}x^2 + \frac{1}{4}x^2 \ln x\,.$$

The constants C_1, C_2 and C_3 are determined by the boundary conditions and can be computed. A sufficiently accurate approximation is given by $C_1 = 2.09512641807804822$, $C_2 = -0.858182223550408451$, and $C_3 = -0.2983567524$

h	EMAX(y)	EMAX(y')	EMAX(y'')
1/8	1.40×10^{-5}	3.13×10^{-5}	6.95×10^{-5}
1/16	2.91×10^{-7}	6.78×10^{-7}	2.21×10^{-6}
1/32	5.00×10^{-9}	1.18×10^{-8}	5.10×10^{-8}
1/40	1.33×10^{-9}	3.13×10^{-9}	8.70×10^{-9}
1/48	4.47×10^{-10}	1.06×10^{-9}	5.14×10^{-9}
1/56	1.71×10^{-10}	4.20×10^{-10}	2.12×10^{-9}
1/64	8.02×10^{-11}	1.89×10^{-10}	9.82×10^{-10}

Table 4: Maximum absolute errors for Example 5.3

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This problem was solved for fixed step-sizes $h = 1/8, 1/16, 1/32, 1/40, 1/48, 1/56, 1/64$. In Table 4, we report the absolute maximum errors for the solution, as well as the absolute maximum errors for the first and second derivatives, which are obtained as $\text{EMAX}(y) = \text{Max}\,|y(x_i) - y_i|$, $\text{EMAX}(y') = \text{Max}\,|y'(x_i) - y_i'|$, and $\text{EMAX}(y'') = \text{Max}\,|y''(x_i) - y_i''|$, $0 = 1, \ldots, N$, where $y(x_i)$ is the exact solution computed at the grid point and y_i is an approximation to the exact solution using MFDMs. It is obvious from the small maximum absolute errors displayed in Table 4 that our method accurately solves third order BVPs subject to mixed boundary conditions. It is also seen from Table 4 that our method exhibits an order 6 behavior, since on halving the step size, EMAX(y) is reduced by a factor $2^6 = 64$.

Example 5.4. Our fourth test problem is the Blasius problem which was taken from Scholarpedia [24].

$$y''' = -\frac{1}{2}yy'', \quad y(0) = 0, \quad y'(0) = 0, \quad y'(\infty) = 1\,.$$

This problem has no theoretical solution, hence the results are presented differently from those solved above. The problem was solved for different values of the truncated boundary x_∞ and a fixed step size $h = 1/8$ was used in all experiments. It is worth noting that our computed values of $y''(0)$ for selected truncated boundaries are in agreement with the values of $y''(0)$ reported in [22]. In particular, our method yielded $y''(0) \approx 0.33206$ at $x_\infty = 8.0$ which is in agreement to 5 decimal digits of accuracy with the result in [22] for $x_\infty = 8.18467$. Details of the numerical results are given in Table 5.

Remark 5.5. Looking at Tables 2 and 3, we deduced that the method in [22] is of order 4, since on halving the step size, EMAX is reduced by a factor

Steps	x_∞	$y''(0)$	$y(x_\infty)$	$y'(x_\infty)$	$y''(x_\infty)$
8	1.0	1.02116	0.50631	1.00000	9.38191×10^{-1}
16	2.0	0.54427	1.05166	1.00000	3.81034×10^{-1}
24	3.0	0.40455	1.67970	1.00000	1.68955×10^{-1}
32	4.0	0.35275	2.43225	1.00000	6.20251×10^{-2}
40	5.0	0.33615	3.31710	1.00000	1.55692×10^{-2}
48	6.0	0.33257	4.28486	1.00000	2.39000×10^{-3}
52	6.5	0.33221	4.78102	1.00000	7.72653×10^{-4}
56	7.0	0.332096	5.27972	1.00000	2.20036×10^{-4}
60	7.5	0.332066	5.77934	1.00000	5.52533×10^{-5}
64	8.0	0.332059	6.27924	1.00000	1.22408×10^{-5}
68	8.5	0.332058	6.77922	1.00000	2.39300×10^{-6}
72	9.0	0.332057	7.27921	1.00000	4.12855×10^{-7}

Table 5: Results obtained for Example 5.4

$2^4 = 16$, while our method exhibits an order 6 behavior, since on halving the step size, EMAX is reduced by a factor $2^6 = 64$. These observations show that our method performs better than the method in [22].

6. Conclusions

We have derived a four-step continuous LMM from which MFDMs are obtained and applied to solve $y''' = f(x, y, y', y'')$ subject to Dirichlet, Neumann, Robin, and free boundary conditions without first adapting the ODE to an equivalent first order system or reducing it to an initial-value problem. The MFDMs are applied as simultaneous numerical integrators over sub-intervals which do not overlap and hence they are more accurate than SFDMs which are generally applied as single formulas over overlapping intervals. We have shown that the methods are convergent and have large intervals of absolute stability, which make them suitable candidates for computing solutions on wider intervals. In addition to providing additional methods and derivatives, the continuous LMM can be used to obtain global error estimates. Our future research will be focused on adapting the MFDMs to solve third order partial differential equations.

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