ON GENERALIZED DIFFERENCE OPERATOR OF THIRD KIND AND ITS APPLICATIONS IN NUMBER THEORY

M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, R. Pugalarasu, S. Elizabeth
1,2,3,4Department of Mathematics
Sacred Heart College
Tirupattur, 635 601, Tamil Nadu, INDIA
e-mail: manuelsmsm03@yahoo.co.in
5Department of Mathematics
Auxilium College
Vellore, Tamil Nadu, INDIA

Abstract: In this paper, the authors extend the theory of the generalized difference operator $\Delta_{\ell}$ and the second kind $\Delta_{\ell_1,\ell_2}$ to the generalized difference operator of the third kind $\Delta_{\ell_1,\ell_2,\ell_3}$ for the positive reals $\ell_1, \ell_2$ and $\ell_3$. We also present the discrete version of Leibnitz Theorem, binomial theorem, Newton’s formula with reference to $\Delta_{\ell_1,\ell_2,\ell_3}$. Also by defining its inverse, we establish a few formulae for the sum of the second partial sums of higher powers of arithmetic progression in number theory.

AMS Subject Classification: 39A

Key Words: generalized difference operator, generalized polynomial factorial, partial sums

1. Introduction

The theory of difference equations is based on the operator $\Delta$ defined as
$$\Delta u(n) = u(n + 1) - u(n), \quad n \in \mathbb{N},$$
where $\mathbb{N} = \{0, 1, 2, 3, \cdots \}$. Eventhough many authors [1], [8], [9] have suggested the definition of $\Delta$ as...
\[ \Delta u(n) = u(n + \ell) - u(n), \quad n \in \mathbb{N}, \quad \ell \in \mathbb{R} - \{0\}, \quad (2) \]

no significant progress took place on this line. But recently, when we took up the definition of \( \Delta \) as given in (2) and developed the theory of difference equations in a different direction and obtained some interesting results in the application of number theory. For convinence we labelled the operator \( \Delta \) defined by (2) as \( \Delta_\ell \) and by defining its inverse \( \Delta_\ell^{-1} \) many interesting results on number theory were obtained. By extending the study for sequences of complex numbers and \( \ell \) to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and weblike were studied for the solutions of difference equations involving \( \Delta_\ell \). The results obtained can be found in [2]-[6]. The goal of this paper is to obtain some significant results on \( \Delta_\ell \), \( \Delta_{\ell_1,\ell_2} \), \( \Delta_{\ell_1,\ell_2,\ell_3} \) and to obtain the values of \( S^n \), \( PS^n \) and \( P^2S^n \) where

\[
S^n = j^n + (j + \ell)^n + (j + 2\ell)^n + \cdots + (j + k\ell)^n,
\]

\[
PS^n = j^n + j^n + (j + \ell)^n + j^n + (j + \ell)^n + (j + 2\ell)^n + \cdots + j^n + (j + \ell)^n + \cdots + (j + k\ell)^n,
\]

\[
P^2S^n = j^n + \left[j^n + j^n + (j + \ell)^n\right] + \left[j^n + j^n + (j + \ell)^n\right] + \cdots + \left[j^n + j^n + (j + \ell)^n + j^n + (j + \ell)^n + (j + 2\ell)^n\right] + \cdots + j^n + (j + \ell)^n + (j + 2\ell)^n + \cdots + (j + k\ell)^n.
\]

\( S^n \) denotes the sum of the \( n \)-th powers of an A.P., \( PS^n \) denotes the sum of the partial sums of \( S^n \) and \( P^2S^n \) denotes sum of partial sums of \( PS^n \).

The formula for the value of \( S^n \), \( PS^n \) are derived in [2], [7] using \( \Delta_\ell \), \( \Delta_{\ell,\ell} \) respectively. Hence, in this paper, we derive the formulae for finding the value of \( P^2S^n \) using \( \Delta_{\ell,\ell,\ell} \), Stirling numbers of the second kind and present here some results and applications on \( \Delta_{\ell_1,\ell_2,\ell_3} \).

Throughout this paper, we make use of the following assumptions:

(i) \( N = \{0, 1, 2, 3, \ldots\} \);
(ii) \( N(a) = \{a, a + 1, a + 2, \ldots\} \),
(iii) \( r \) and \( n \) are positive integers and \( \ell_1, \ell_2 \) and \( \ell_3 \) are positive reals,
(iv) \( n^* \) is the largest non negative integer such that \( k - n^*\ell \geq 0 \),
(v) \( c, c_0, c_1, c_2, \ldots \) are constants,
(vi) \( rC_i = \frac{r!}{(r-i)!i!} \) where \( 0! = 1, \ r! = 1.2.3\ldots r \) and
(vii) \( u : [0, \infty) \to \mathbb{C} \) is a complex valued function on \( [0, \infty) \).
2. Basic Definitions and Examples

In this section, we present the discrete versions of Leibnitz and binomial theorems with reference to \( \Delta_{\ell_1,\ell_2,\ell_3} \). Suitable examples are provided to illustrate the results.

**Definition 2.1.** Let \( u : [0, \infty) \rightarrow \mathbb{C} \) be any complex valued function on \( [0, \infty) \). We define the generalized difference operator of the third kind for \( u(k) \) as
\[
\Delta_{\ell_1,\ell_2,\ell_3} u(k) = u(k + \ell_1 + \ell_2 + \ell_3) - [u(k + \ell_1 + \ell_2) + u(k + \ell_1 + \ell_3) + u(k + \ell_2 + \ell_3)] + [u(k + \ell_1) + u(k + \ell_2) + u(k + \ell_3)] - u(k).
\]

**Lemma 2.2.** If \( E^\ell \) is the usual shift operator defined as \( E^\ell u(k) = u(k+\ell) \), then the following are simple to derive. If \( \ell_j, j = 1, 2, 3 \) are positive reals, then:
\[
(i) \quad \Delta_{\ell_1,\ell_2,\ell_3} = E^{\ell_1+\ell_2+\ell_3} - (E^{\ell_1+\ell_2} + E^{\ell_1+\ell_3} + E^{\ell_2+\ell_3}) - 1,
\]
\[
(ii) \quad \Delta_{\ell_1,\ell_2,\ell_3} = \Delta_{\ell_1+\ell_2+\ell_3} - (\Delta_{\ell_1+\ell_2} + \Delta_{\ell_1+\ell_3} + \Delta_{\ell_2+\ell_3}) + (\Delta_{\ell_1} + \Delta_{\ell_2} + \Delta_{\ell_3}),
\]
\[
(iii) \quad \Delta_{\ell_1,\ell_2,\ell_3} = \Delta_{\ell_1} \Delta_{\ell_2} \Delta_{\ell_3}, \quad \text{and}
\]
\[
(iv) \quad \Delta_{\ell_1,\ell_2,\ell_3} = \prod_{j=1}^{3} \left( \sum_{i=1}^{\ell_j} \ell_j C_i \Delta^{i} \right).
\]

**Definition 2.3.** The second order of the generalized difference operator of the third kind is \( \Delta_{\ell_1,\ell_2,\ell_3}^2 = \Delta_{\ell_1,\ell_2,\ell_3} (\Delta_{\ell_1,\ell_2,\ell_3}) \) and in general the \( n \)-th order of the generalized difference operator of the third kind is defined as \( \Delta_{\ell_1,\ell_2,\ell_3}^n = \Delta_{\ell_1,\ell_2,\ell_3} (\Delta_{\ell_1,\ell_2,\ell_3}^{n-1}) \).

**Remark 2.4.** For the positive integers \( p \) and \( q \),
\[
\Delta_{\ell_1,\ell_2,\ell_3}^p \Delta_{\ell_1,\ell_2,\ell_3}^q = \Delta_{\ell_1,\ell_2,\ell_3}^q \Delta_{\ell_1,\ell_2,\ell_3}^p.
\]

As a consequence of Definition 2.3, the following results can be obtained easily.

**Lemma 2.5.** (i) If \( P_{3p-1}(k) = c_{3p-1}k^{3p-1} + c_{3p-2}k^{3p-2} + \cdots + c_1k + c_0 \) is any polynomial in \( k \) of degree \( (3p - 1) \), then \( \Delta_{\ell_1,\ell_2,\ell_3}^p P_{3p-1}(k) = 0 \).
(ii) If \( m \) and \( n \) are positive integers and \( \ell \) is a real number, then
\[
\Delta_{\ell,\ell}^n k^m = \begin{cases} 
  m! \ell^m, & \text{if } m = 3n; \\
  0, & \text{if } m < 3n.
\end{cases}
\]  

(iii) If \( P_k = a_0 k^{3n} + a_1 k^{3n-1} + a_2 k^{3n-2} + \cdots + a_n \) is any polynomial in \( k \) of degree \( 3n \), then
\[
\Delta_{\ell,\ell}^n P_k = a_0 (3n!) \ell^{3n}.
\]  

(iv) For the positive integer \( r \),
\[
\Delta_{\ell_1,\ell_2,\ell_3}^r = \prod_{j=1}^{3} \left( \sum_{i=0}^{r} (-1)^i r C_i (r-i) \right)
\]  

which is equivalent to
\[
\Delta_{\ell_1,\ell_2,\ell_3}^r u(k) = \prod_{j=1}^{3} \left( \sum_{i=0}^{r} (-1)^i r C_i (k + \ell_j (r-i)) \right).
\]  

(v) If \( \ell_j = \sum_{i=1}^{n} \ell_{j_i}, j = 1, 2, 3 \), then \( \Delta_{\ell_1,\ell_2,\ell_3} = \prod_{j=1}^{3} \left[ \prod_{i=1}^{n} \left( \Delta_{\ell_{j_i}} + 1 \right) - 1 \right] \).

(vi) For the positive integer \( n \),
\[
\Delta_{n\ell_1,n\ell_2,n\ell_3} = E^n(\ell_{1}+\ell_{2}+\ell_{3}) - (E^n(\ell_{1}+\ell_{2}) + E^n(\ell_{1}+\ell_{3}) + E^n(\ell_{2}+\ell_{3}))
\]  

\[+ (E^n\ell_1 + E^n\ell_2 + E^n\ell_3) - 1,
\]  

\[
\Delta_{n\ell_1,n\ell_2,n\ell_3} = (1 + \Delta_{\ell_{1}+\ell_{2}+\ell_{3}})^n - [(1 + \Delta_{\ell_{1}+\ell_{2}})^n + (1 + \Delta_{\ell_{1}+\ell_{3}})^n + (1 + \Delta_{\ell_{2}+\ell_{3}})^n - 1, \]
\[
\Delta_{n\ell_1,n\ell_2,n\ell_3} = \sum_{r=1}^{n} n C_r \{ \Delta_{\ell_{1}+\ell_{2}+\ell_{3}}^r - (\Delta_{\ell_{1}+\ell_{2}}^r + \Delta_{\ell_{1}+\ell_{3}}^r + \Delta_{\ell_{2}+\ell_{3}}^r)
\]  

\[+ (\Delta_{\ell_{1}}^r + \Delta_{\ell_{2}}^r + \Delta_{\ell_{3}}^r) \},
\]  

\[
\Delta_{n\ell_1,n\ell_2} = \sum_{r=0}^{n} (-1)^r n C_r \Delta_{\ell_{1}+\ell_{2}+\ell_{3}}^{n-r}
\]  

\[
\left\{ \sum_{i=0}^{r} (-1)^r r C_i (\Delta_{\ell_{1}+\ell_{2}} + \Delta_{\ell_{1}+\ell_{3}} + \Delta_{\ell_{2}+\ell_{3}})^{r-i} (\Delta_{\ell_{1}} + \Delta_{\ell_{2}} + \Delta_{\ell_{3}})^i \right\},
\]  

\[
\Delta_{n\ell_1,n\ell_2} = \prod_{j=1}^{3} \left( \sum_{i=0}^{n-1} (-1)^i n C_i (\Delta_{n-i}\ell_{j}) \right).
\]

The following is the discrete version of Leibnitz Theorem according to \( \Delta_{\ell_1,\ell_2,\ell_3} \).
Theorem 2.6. For the functions \( u : [0, \infty) \to \mathbb{C}, v : [0, \infty) \to \mathbb{C}, \)
\[
\Delta_{\ell_1, \ell_2, \ell_3}^n[u(k)v(k)] = \Delta_{\ell_1}^n \left( \Delta_{\ell_2}^n \left[ u(k) \Delta_{\ell_3}^n v(k) \right] \right) + nC_1 \Delta_{\ell_1}^n \left( \Delta_{\ell_2}^n \left[ \Delta_{\ell_3}^n u(k) \right] \right) \\
\Delta_{\ell_3}^{n+1}[v(k + \ell_3)] + \cdots + nC_n \Delta_{\ell_1}^n \left( \Delta_{\ell_2}^n \left[ \Delta_{\ell_3}^n u(k) v(k + n\ell_3) \right] \right). \tag{17}
\]

Proof. The proof follows from the generalized Leibnitz Theorem (Theorem 2.5 [2]) and (6).

Lemma 2.7. If \( n \) is a positive integer, then
\[
E^n(\ell_1+\ell_2+\ell_3) - (E^n(\ell_1+\ell_2) + E^n(\ell_1+\ell_3) + E^n(\ell_2+\ell_3)) + (E^{n\ell_1} + E^{n\ell_2} + E^{n\ell_3}) \\
= \sum_{r=1}^n nC_r \{ \sum_{i=0}^{r-1} (-1)^i rC_i \left[ \Delta_{(r-i)\ell_1+\ell_2+\ell_3}^n - (\Delta_{(r-i)\ell_1+\ell_2}^n + \Delta_{(r-i)\ell_1+\ell_3}^n + \Delta_{(r-i)\ell_2+\ell_3}^n) \right] \\
+ \Delta_{(r-i)\ell_1} + \Delta_{(r-i)\ell_2} + \Delta_{(r-i)\ell_3} \}. \tag{18}
\]

Theorem 2.8. If \( n \) and \( p \) are the positive integers, then
\[
(k + n(\ell_1 + \ell_2 + \ell_3))^{p} - [(k + n(\ell_1 + \ell_2))^{p} + (k + n(\ell_1 + \ell_3))^{p} + (k + n(\ell_2 + \ell_3))^{p}] + [(k + n\ell_1)^{p} + (k + \ell_1\ell_2)^{p} + (k + n\ell_2)^{p} + (k + \ell_2\ell_3)^{p}] - rC_1 \left[ (k + (r - 1)\ell_1(\ell_1 + \ell_2 + \ell_3))^{p} - (k + r\ell_1)^{p} \\
+ (k + (r - 1)\ell_1\ell_2)^{p} + (k + (r - 1)\ell_1\ell_3)^{p} + (k + (r - 1)\ell_2\ell_3)^{p} - k^{p} \right] + \cdots + \\
+ (-1)^{r-1} rC_{r-1} \left[ (k + \ell_1 + \ell_2 + \ell_3)^{p} - (k + \ell_1\ell_2)^{p} + (k + \ell_1\ell_3)^{p} \right] \\
+ (k + \ell_2 + \ell_3)^{p} + [(k + \ell_1)^{p} + (k + \ell_2)^{p} + (k + \ell_3)^{p} - k^{p}] \right]. \tag{19}
\]

Proof. The proof follows by operating (18) on \( u(k) = k^{p} \).

Example 2.9. If \( \theta_1, \theta_2 \) and \( \theta_3 \) are angles measured along the anticlockwise direction, then
\[
\sin(k + n(\theta_1 + \theta_2 + \theta_3)) - [\sin(k + n(\theta_1 + \theta_2)) + \sin(k + n(\theta_1 + \theta_3)) \\
+ \sin(k + n(\theta_2 + \theta_3))] + [\sin(k + n\theta_1) + \sin(k + n\theta_2) + \sin(k + n\theta_3)] \\
= \sum_{r=1}^n nC_r \left[ (k + r\theta_1 + \theta_2 + \theta_3)^{p} - (k + r(\theta_1 + \theta_2))^{p} \\
+ \sin(k + r\theta_1) \right] + [\sin(k + r\theta_2) + \sin(k + r\theta_3)] - sin k] - rC_1 \left[ (k + (r - 1)(\theta_1 + \theta_2 + \theta_3)) \\
- [\sin(k + (r - 1)(\theta_1 + \theta_2)) + \sin(k + (r - 1)(\theta_1 + \theta_3)) + \sin(k + (r - 1)(\theta_2 + \theta_3))] \\
+ [\sin(k + (r - 1)\theta_1) + \sin(k + (r - 1)\theta_2) + \sin(k + (r - 1)\theta_3)] - sin k] + \cdots +
\]
\[ + (-1)^{r-1} r C_{r-1} \sin(k + \theta_1 + \theta_2 + \theta_3) - [\sin(k + \theta_1 + \theta_2) + \sin(k + \theta_1 + \theta_3) + \sin(k + \theta_2 + \theta_3)] + [\sin(k + \theta_1) + \sin(k + \theta_2) + \sin(k + \theta_3)] - \sin k].\]

**Lemma 2.10.** Let \( u : [0, \infty) \rightarrow \mathbb{C} \) is a function and \( x \) is real. Then

\[
\sum_{j=0}^{\infty} \left\{ \frac{x^j(\ell_1 + \ell_2 + \ell_3)}{j!(\ell_1 + \ell_2 + \ell_3)^j} u(j(\ell_1 + \ell_2 + \ell_3)) - \frac{x^j(\ell_1 + \ell_2)}{j!(\ell_1 + \ell_2)^j} u(j(\ell_1 + \ell_2))\right\}
\]

\[
+ \frac{x^j(\ell_1 + \ell_3)}{j!(\ell_1 + \ell_3)^j} u(j(\ell_1 + \ell_3)) + \frac{x^j(\ell_2 + \ell_3)}{j!(\ell_2 + \ell_3)^j} u(j(\ell_2 + \ell_3))\right\}
\]

\[
+ \frac{x^j}{j!} u(j(\ell_3)) \right\} = \left\{ e^{-x^{(\ell_1 + \ell_2 + \ell_3)}} e^{x^{(\ell_1 + \ell_2 + \ell_3)} u(0)}\right\} u(0).
\]

**Proof.** The proof follows from (4), \( u(k) = E^k u(0) \) and \( E^\ell = 1 + \Delta \ell \). \( \square \)

**Corollary 2.11.** If \( \ell \) is a positive real, then

\[
\sum_{j=0}^{\infty} \left\{ \frac{x^j(3\ell)}{j!(3\ell)^j} u(j(3\ell)) - \frac{x^j(2\ell)}{j!(2\ell)^j} u(j(2\ell)) + \frac{x^j}{j!} u(j(\ell))\right\}
\]

\[
= \left\{ e^{x^3/3} - 3e^{x^2/2} + 3e^{x/6} \right\} u(0)
\]

\[
= \left\{ e^{\Delta x/3} - 3e^{\Delta x/2} + 3e^{\Delta x/6} \right\} u(0).
\]

### 3. Generalized Polynomial Factorial of the Third Kind

In this section, we establish the relation between the generalized polynomial factorial and polynomial and discrete version of Newton’s formula on \( \Delta_{\ell,\ell,\ell} \).
Definition 3.1. For the positive integer $n$, the generalized polynomial factorial in $k$ of the third kind is defined as
\[ k_{t_1,t_2,t_3}^{(n)} = (k + \ell_2 + \ell_3)_{t_1}^{(n)} + (k + \ell_1 + \ell_3)_{t_2}^{(n)} + (k + \ell_1 + \ell_2)_{t_3}^{(n)} \]
\[ - \{ (k + \ell_2)_{t_1}^{(n)} + (k + \ell_3)_{t_2}^{(n)} + (k + \ell_1)_{t_3}^{(n)} \}
\[ + (k + \ell_3)_{t_2}^{(n)} + (k + \ell_1)_{t_3}^{(n)} + (k + \ell_2)_{t_3}^{(n)} \} + k_{t_1}^{(n)} + k_{t_2}^{(n)} + k_{t_3}^{(n)}. \quad (20) \]

Using the Stirling numbers of the first kind $s_r^n$, the following can be easily obtained.

Lemma 3.2. If $n$ is any positive integer and any real $t$, then
\[ \sum_{r=1}^{n} s_r^n t^{n-r} \Delta_{t_1,t_2,t_3} k_t^{(n)} = \Delta_{t_1,t_2,t_3} k_t^{(n)} \quad (21) \]
and
\[ \Delta_{t_1,t_2,t_3} k_t^{(n)} = \sum_{r=0}^{m} (-1)^r m C_r \sum_{i=1}^{n} s_i^n t^{r-i} \left\{ \sum_{j=0}^{m} (-1)^j m C_j \right\} \quad (22) \]

Proof. The proof follows from (6) and the relation
\[ \Delta_k^{m} k_t^{(n)} = \sum_{r=0}^{m} [(-1)^r m C_r \sum_{i=1}^{n} s_i^n t^{r-i} (k + (m - r) \ell)^i]. \]

Lemma 3.3. If $n$ is a positive integer, then $\Delta_{t_1,t_2,t_3} k_t^{(n)}$ is
\[ \begin{cases} 
  n \ell_1 [(k + \ell_2 + \ell_3)_{t_1}^{(n-1)} - (k + \ell_2)_{t_1}^{(n-1)} + (k + \ell_3)_{t_1}^{(n-1)}], & \text{if } t = t_1; \\
  n \ell_2 [(k + \ell_1 + \ell_3)_{t_2}^{(n-1)} - (k + \ell_1)_{t_2}^{(n-1)} + (k + \ell_3)_{t_2}^{(n-1)}], & \text{if } t = t_2; \\
  n \ell_3 [(k + \ell_1 + \ell_2)_{t_3}^{(n-1)} - (k + \ell_1)_{t_3}^{(n-1)} + (k + \ell_2)_{t_3}^{(n-1)}], & \text{if } t = t_3. 
\end{cases} \quad (23) \]

Proof. The proof follows from (3), (20) and $k_t^{(n-1)}$. \[ \square \]

Lemma 3.4. If $n$ is a positive integer, then
\[ \Delta_{t_1,t_2,t_3} k_{t_1,t_2,t_3}^{(n)} = n \ell_1 \Delta_{t_2,t_3} k_{t_1}^{(n-1)} + n \ell_2 \Delta_{t_1,t_3} k_{t_2}^{(n-1)} + n \ell_3 \Delta_{t_1,t_2} k_{t_3}^{(n-1)}. \quad (24) \]
Proof. The proof follows from (6) and $\Delta_k k^{(n)} = n k^{(n-1)}$. \hfill \Box

**Corollary 3.5.** Let $n$ be a positive integer. Then

$$\Delta_{\ell, \ell, \ell} k^{(n)} = (n \ell)_\ell k^{(n-3)}.$$  \hfill (25)

The following theorem is the generalized version of Newton’s formula with reference to $\Delta_{\ell, \ell, \ell}$.

**Theorem 3.6.** Let $f(k)$ be a polynomial in $k$ of degree $3n$. Then $f(k)$ can be expressed as

$$f(k) = f(0) + \Delta_{\ell, \ell, \ell} f(0) k^{(3)} + \frac{\Delta^2_{\ell, \ell, \ell} f(0)}{6!} k^{(6)} + \cdots + \frac{\Delta^n_{\ell, \ell, \ell} f(0)}{(3n)!} k^{(3n)}.$$  \hfill (26)

Proof. Assume that

$$f(k) = a_0 + a_1 k^{(3)} + a_2 k^{(6)} + \cdots + a_n k^{(3n)}.$$  \hfill (27)

The coefficients are determined from the relation

$$\Delta_{\ell, \ell, \ell} f(0) = a_r (3r)! \ell^{3r}.$$  \hfill (28)

The rest of the proof follows from (27) and (28). \hfill \Box

**Corollary 3.7.** Let $f(k)$ be a polynomial in $k$ of degree $3n$. Then $f(k-t)$ can be expressed as

$$f(t) + \Delta_{\ell, \ell, \ell} f(t) (k-t)^{(3)} + \frac{\Delta^2_{\ell, \ell, \ell} f(t)}{6!} (k-t)^{(6)} + \cdots + \frac{\Delta^n_{\ell, \ell, \ell} f(t)}{(3n)!} (k-t)^{(3n)}.$$  \hfill (29)

Proof. Replacing $0$ by $t$ and $k$ by $(k-t)$ in (26) we obtain the result as desired. \hfill \Box

4. Inverse of Generalized Difference Operator of the Third Kind and its Applications

In this section, we define the inverse $\Delta^{-1}_{\ell_1, \ell_2, \ell_3}$ and present some results using the inverse which will be used to find $P^2 S^n$.

**Definition 4.1.** The inverse of generalized difference operator of the third kind denoted by $\Delta^{-1}_{\ell_1, \ell_2, \ell_3}$ is defined as follows.
If $\Delta_{\ell_1,\ell_2,\ell_3}z(k) = y(k)$, then
\[ z(k) = \Delta^{-1}_{\ell_1,\ell_2,\ell_3}y(k) + c_{2j}\left(\frac{k(2)_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k(1)_{\ell_1}}{\ell_1}\right) + c_{0j}, \tag{30} \]
respectively, where $c_{0j}, c_{1j}$ and $c_{2j}$’s are constants which depend on $k - n^*\ell$.

**Lemma 4.2.** If $n$ is a positive integer and $k \in [n\ell, \infty)$, then
\[ \Delta^{-1}_{\ell_1,\ell_2,\ell_3}(k^{(n)}_{\ell_1,\ell_2,\ell_3}) = \frac{k^{(n+1)}_{\ell_1}}{\ell_1(n+1)} + \frac{k^{(n+1)}_{\ell_2}}{\ell_2(n+1)} + \frac{k^{(n+1)}_{\ell_3}}{\ell_3(n+1)} + c_{2j}\left(\frac{k(2)_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k(1)_{\ell_1}}{\ell_1}\right) + c_{0j}, \tag{31} \]
and
\[ \Delta^{-1}_{\ell,\ell,\ell}(k^{(n)}_{\ell,\ell,\ell}) = 3\frac{k^{(n+1)}_{\ell}}{\ell(n+1)} + c_{2j}\left(\frac{k(2)_{\ell}}{2\ell^2}\right) + c_{1j}\left(\frac{k(1)_{\ell}}{\ell}\right) + c_{0j}. \tag{32} \]

**Proof.** The proof follows from (30) and $\Delta_{\ell}(k^{(n+1)}_{\ell}) + c = \ell(n+1)k^{(n)}_{\ell}$. \(\square\)

**Theorem 4.3.** There exits constants $c_{0j}, c_{1j}$ and $c_{2j}$ which depend on $k - n^*\ell$ such that
\[ \Delta^{-1}_{\ell,\ell,\ell}y(k) = \sum_{t=2}^{n^*} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} y(k - t\ell - s\ell - r\ell) + c_{2j}\left(\frac{k(2)_{\ell}}{2\ell^2}\right) + c_{1j}\left(\frac{k_{\ell}}{\ell}\right) + c_{0j}. \tag{33} \]

**Proof.** The proof follows by the relation $\Delta_{\ell,\ell,\ell}\left\{ \sum_{t=2}^{n^*} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} y(k - t\ell - s\ell - r\ell) + c_{2j}\left(\frac{k(2)_{\ell}}{2\ell^2}\right) + c_{1j}\left(\frac{k_{\ell}}{\ell}\right) + c_{0j} \right\} = y(k)$. \(\square\)

**Lemma 4.4.** If $\lambda \neq 1, k \geq 3\ell$ and $P_k$ is any function of $k$, then
\[ \sum_{t=2}^{n^*} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} \lambda^{k-t\ell-s\ell-r\ell} P_{k-t\ell-s\ell-r\ell} = \lambda^k \left(1 - \frac{\lambda^r \Delta_{\ell}}{(\lambda^r - 1)^3} \right) \left\{ 1 - \frac{\lambda^r \Delta_{\ell}}{(\lambda^r - 1)} \right\} + \frac{\lambda^{2r \Delta_{\ell}^2}}{(\lambda^r - 1)^2} \cdots \right\}^3 P_k + c_{2j}\left(\frac{k(2)_{\ell}}{2\ell^2}\right) + c_{1j}\left(\frac{k_{\ell}}{\ell}\right) + c_{0j}. \tag{34} \]
Proof. Let $\Delta_{\ell,\ell,\ell} \lambda^k F_k = \lambda^k P_k$, where $P_k = (\lambda \ell^E - 1)^3 F_k$. Operating both sides by $\Delta_{\ell,\ell,\ell}^{-1}$, we obtain

$$\Delta_{\ell,\ell,\ell}^{-1} \lambda^k P_k = \lambda^k F_k + c_{2j} \left( \frac{k_{\ell}^{(2)}}{2\ell^2} \right) + c_{1j} \left( \frac{k}{\ell} \right) + c_{0j}$$

$$= \lambda^k (\lambda \ell^E - 1)^{-3} P_k + c_{2j} \left( \frac{k_{\ell}^{(2)}}{2\ell^2} \right) + c_{1j} \left( \frac{k}{\ell} \right) + c_{0j}.$$  

The rest of the proof follows from (33) and the binomial theorem. \hfill \Box

**Lemma 4.5.** The relation between $\Delta_{\ell,\ell,\ell}^{-1}$ and $\Delta_{\ell}^{-1}$ is

$$\sum_{p=0}^{\ell-1} \sum_{q=0}^{\ell-1} \sum_{r=0}^{\ell-1} \Delta_{\ell,\ell,\ell}^{-1} u(k + p + q + r) = \Delta_{\ell}^{-1} (\Delta_{\ell}^{-1} u(k))$$

$$+ c_{2j} \left( \frac{k_{\ell}^{(2)}}{2\ell^2} \right) + c_{1j} \left( \frac{k}{\ell} \right) + c_{0j}.$$  

Proof. The proof follows from $\sum_{i=0}^{\ell-1} \Delta_{\ell}^{-1} u(k + i) = \Delta_{\ell}^{-1} u(k) + c$ and (33). \hfill \Box

The following two lemmas are easy deductions.

**Lemma 4.6.** If $S^n_r$’s are the Stirling numbers of the second kind, then

$$(k + 2\ell)^n - 2(k + \ell)^n + k^n = \frac{1}{3} \sum_{r=1}^{n} S^n_r (n^{-r}) k_{\ell,\ell,\ell}^{(r)}.$$  

(35)

**Lemma 4.7.** If $n$ is a positive integer, then

(i) $\Delta_{\ell,\ell,\ell}^{-1} k_{\ell}^{(n)} = \frac{k_{\ell,\ell,\ell}^{(n+5)}}{3n(n+1)\cdots(n+5)\ell^5} + c_{2j} \left( \frac{k_{\ell}^{(2)}}{2\ell^2} \right) + c_{1j} \left( \frac{k}{\ell} \right) + c_{0j},$  

(36)

(ii) $k^n = \frac{1}{3} \sum_{r=1}^{n} S^n_r (n^{-r}) \Delta_{\ell,\ell,\ell}^{-1} k_{\ell,\ell,\ell}^{(r)}.$  

(37)

**Lemma 4.8.** If $\ell$ is any positive real number and $k \in [4\ell, \infty)$, then

$$\sum_{t=2}^{n^*} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} (k - t\ell - s\ell - r\ell)^3 + c_{2j} \left( \frac{k_{\ell}^{(2)}}{2\ell^2} \right) + c_{1j} \left( \frac{k}{\ell} \right) + c_{0j}$$

$$= \frac{k_{\ell}^{(6)}}{120\ell^3} + \frac{k_{\ell}^{(5)}}{20\ell^2} + \frac{k_{\ell}^{(4)}}{24\ell}.$$  

(38)
Proof. The proof follows from (33), (6) and $k^n = \sum_{r=1}^{n} S^r_{\ell} \ell^{n-r} k_{\ell}^{(r)}$ (see [2]).

The following theorem is the general rule to find the value of $P^2 S^n$, where $S^n$ is the sum of $n$-th powers of an arithmetic progression.

**Theorem 4.9.** If $S^n_k$’s are the Stirling numbers of second kind, then
\[
\sum_{t=2}^{n^*} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} (k - tl - sl - rl)^n
= \sum_{t=1}^{n} \frac{S^n_{\ell} \ell^{n(t+3)}}{(t+1)(t+2)(t+3)} \left\{ k_{\ell}^{(t+3)} - \left( (n + 2)\ell + j \right)^{(t+3)}_\ell \right\}
+ \sum_{t=2}^{n^*} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} (n + 3)\ell - tl - sl - rl + i)^n
+ \left( (n + 2) + \frac{(j - k)}{\ell} \right)

\left\{ \sum_{t=1}^{n} \frac{S^n_{\ell} \ell^{n(t+3)}}{(t+1)(t+2)(t+3)} \left[ ((n + 3)\ell + j)^{(t+3)}_\ell - ((n + 2)\ell + j)^{(t+3)}_\ell \right] - \sum_{t=2}^{n^*} (n + 3)\ell - tl + j)^n \right\} + \frac{1}{2e^2} \left[ ((n + 2)\ell + j)(2k - ((n - 3)\ell + j)) \right]

\left\{ \sum_{t=1}^{n} \frac{S^n_{\ell} \ell^{n(t+3)}}{(t+1)(t+2)(t+3)} \left[ ((n + 4)\ell + j)^{(t+3)}_\ell - ((n + 2)\ell + j)^{(t+3)}_\ell \right] - \sum_{t=2}^{n^*} (n + 3)\ell - tl + j)^n \right\} = P^2 S^n. \quad (39)

Proof. From (38), we obtain
\[
\sum_{t=2}^{n^*} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} (k - tl - sl - rl)^n + c_{2j} \left( k_{\ell}^{(2)} \right) + c_{1j} \left( \frac{k}{\ell} \right) + c_{0j}
\]

\[
= \sum_{t=1}^{n} \frac{S^n_{\ell} \ell^{n(t+3)}}{(t+1)(t+2)(t+3)} k_{\ell}^{(t+3)}. \quad (40)
\]
Replacing \( k \) by \((n + 2)\ell + j\), we get
\[
c_{0j} = \sum_{t=1}^{n} \frac{S_{t}^{n} p_{n-(t+3)}}{(t+1)(t+2)(t+3)}( (n + 2)\ell + j)^{(t+3)} - \sum_{s=2}^{n} \sum_{r=0}^{n^*} ( (n + 2)\ell - s\ell - r\ell + j)^{n}.
\]

Substituting (41) in (40) and replacing \( k \) by \((n + 3)\ell + j\), we obtain
\[
c_{1j} = \sum_{t=1}^{n} \frac{S_{t}^{n} p_{n-(t+3)}}{(t+1)(t+2)(t+3)}[ ( (n + 2)\ell + j)^{(t+3)} - ( (n + 1)\ell + j)^{(t+2)} ]
\]
\[
+ \frac{c_{2j}}{2\ell^2} [ ( (n + 2)\ell + j)^{(2)} + ( (n + 3)\ell + j)^{(2)} ] - \sum_{s=2}^{n^*} ( (n + 3)\ell - t\ell - s\ell + j)^{n}. \tag{42}
\]

Substituting (42) and (41) in (40) and replacing \( k \) by \((n + 4)\ell + j\), we obtain
\[
c_{2j} = \sum_{t=1}^{n} \frac{S_{t}^{n} p_{n-(t+3)}}{(t+1)(t+2)(t+3)}[ ( (n + 4)\ell + j)^{(t+3)} - ( (n + 2)\ell + i)^{(t+2)} ]
\]
\[
+ \sum_{t=2}^{n} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} ( (n + 2)\ell - t\ell - s\ell - r\ell + j)^{n}
\]
\[
- 2 \left\{ \sum_{t=1}^{n} \frac{S_{t}^{n} p_{n-(t+3)}}{(t+1)(t+2)(t+3)}[ ( (n + 3)\ell + j)^{(t+3)} - ( (n + 2)\ell + i)^{(t+2)} ]
\right\}
\]
\[
- \sum_{t=2}^{n} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} ( (n + 3)\ell - t\ell - s\ell + j)^{n} \} - \sum_{t=2}^{n} \sum_{s=1}^{n^*} \sum_{r=0}^{n^*} ( (n + 4)\ell - t\ell - s\ell - r\ell + j)^{n}. \tag{43}
\]

The proof follows from (40), (41), (42) and (43). \(\square\)

**Example 4.10.** For the A.P. 3, 10, 17, \(\ldots\), 101, we shall find \(S^5\), \(PS^5\) and \(P^2S^5\) where \(S^5 = 3^5 + 10^5 + 17^5 + \cdots + 101^5\), the sum of fifth powers of the A.P., \(PS^5 = 3^5 + 3^5 + 10^5 + 10^5 + 17^5 + \cdots + 3^5 + 10^5 + 17^5 + \cdots + 101^5\), the sum of all partial sums from \(S^5\) and \(P^2S^5 = 3^5 + (3^5 + 3^5 + 10^5) + (3^5 + 3^5 + 10^5 + 3^5 + 10^5 + 17^5) + \cdots + (3^5 + 3^5 + 10^5 + 3^5 + 10^5 + 17^5 + \cdots + 101^5)\), the sum of all partial sums from \(PS^5\).

Solution. Taking \(n = 5\), \(j = 3\), \(\ell = 7\) and \(k = 122\) in Theorem 4.9, we obtain \(P^2S^5 = 201550764800\). Similarly we can find the value of \(PS^n\) and \(S^n\) (refer [2], [7]).
Acknowledgements

This research is supported by University Grants Commission, New Delhi.

References


