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COINCIDENCE POINT THEOREMS FOR CONTRACTIVE TYPE OF MULTIVALUED MAPPINGS IN COMPACT METRIC SPACES

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Abstract: In this paper, we establish some new coincidence point theorems for contractive type of multivalued and single valued mappings in compact metric spaces, which extend properly the corresponding results of Hu and Rosen [1], Liu [10] and Rao [11].

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1. Introduction

Many researchers studied the existence of nonunique fixed and coincidence

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points for some classes of contractive type of multivalued mappings in metric spaces (see [1]-[11]). Now we listed three classes of contractive type of multivalued mappings investigated in [1], [10], [11].

$$H(Gx, Gy) < d(x, y) \tag{1.1}$$

for all $x, y \in X$ with $x \neq y$;

$$H(Gx, Gy) < \max\left\{d(fx, fy), d(fx, Gx), d(fy, Gy), \frac{1}{2}\left[d(fx, Gy) + d(fy, Gx)\right]\right\} \tag{1.2}$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx \overline{\in} Gx$, $fy \overline{\in} Gy$;

$$H(Gx, Gy) < \max\left\{d(fx, fy), d(fx, Gx), d(fy, Gy), \frac{1}{2}\left[d(fx, Gy) + d(fy, Gx)\right], \frac{d(fx, Gx)d(fy, Gy)}{d(fx, fy)}, \frac{d(fx, Gy)d(fy, Gx)}{d(fx, fy)}\right\} \tag{1.3}$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx \overline{\in} Gx$, $fy \overline{\in} Gy$.

In 1982, Hu and Rosen [1] established a fixed point theorem for contractive type of multivalued mappings (1.1). In 1987, Rao [11] provided some coincidence theorems for contractive type of multivalued and single valued mappings (1.2). In 2004, Liu [10] investigated the existence of coincidence point for contractive type of multivalued and single valued mappings (1.3). For more details, we refer the readers to [1]-[11] and the references therein.

Inspired and motivated by the above results, some new coincidence point theorems concerning contractive type of multivalued and single valued mappings in compact metric spaces are presented in this paper, which extend properly the corresponding results of Hu and Rosen [1], Liu [10] and Rao [11].

Let (X, d) be a metric space and $\mathbb{N}$ be the set of all positive integers. $CL(X)$, $CB(X)$ and $C(X)$ denote the families of all nonempty closed, nonempty closed bounded, nonempty compact subsets of X, respectively, and H denotes the Hausdorff metric on $CB(X)$ induced by the metric d on X. That is,

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\} \quad \text{for } A, B \in CB(X),$$

where $d(a, B) = \inf_{b \in B} d(a, b)$. It is obvious that $CL(X) = CB(X) = C(X)$ if (X, d) is a compact metric space. Let $f : X \to X$ be a single valued mapping, T and $G : X \to CL(X)$ be multivalued mappings. f and G are said to be *commutative* (respectively, *strongly commutative*) if $fGx \subseteq Gfx$ (respectively, $Gfx \subseteq fGx$) for all $x \in X$. For any subset $A \subseteq X$, put $GA = \bigcup_{a \in A} Ga$. The composition of G and T is defined by $TGx = T(Gx)$ for $x \in X$. A point $z \in X$ is said to be a *coincidence point* of f and G if $fz \in Gz$.

Lemma 1.1. (see [1]) *Let (X,d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let all powers of G map X into $C(X)$ and G^m be continuous for some m in $\mathbb{N}$. Let $A = \bigcap_{n\in\mathbb{N}} G^n X$. Then A is a nonempty compact subset of X and $GA = A$.*

2. Coincidence Point Theorems

Our main results are as follows.

Theorem 2.1. *Let (X,d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let all powers of fG map X into $C(X)$ and f, G and $(fG)^m$ be continuous, where m is some element in $\mathbb{N}$. Suppose that*

$$f \text{ and } G \text{ are commutative,} \tag{2.1}$$

$$G(fG)x \subseteq (fG)Gx \quad \text{for all } x \in X \tag{2.2}$$

and

$$\begin{aligned} H(Gx,Gy) &- \min\{d(fx,Gy), d(fy,Gx)\} \\ &< \max\Big\{d(fx,Gx), d(fy,Gy), d(fx,fy), d(fy,Gx), \\ &\quad \frac{1}{2}\big[d(fx,Gy)+d(fy,Gx)\big], \frac{1}{2}\big[H(Gx,Gy)+d(fx,Gx)\big], \\ &\quad \frac{1}{2}\big[H(Gx,Gy)+d(fy,Gy)\big], \frac{1}{2}\big[H(Gx,Gy)+d(fx,fy)\big], \\ &\quad \frac{d(fx,Gy)d(fy,Gx)}{H(Gx,Gy)}, \frac{d(fx,Gx)d(fy,Gy)}{d(fx,fy)}, \\ &\quad \frac{d(fx,Gy)d(fy,Gx)}{d(fx,fy)}, \frac{H(Gx,Gy)d(fx,Gx)}{d(fx,fy)}\Big\} \end{aligned} \tag{2.3}$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx \overline{\in} Gx$ and $fy \overline{\in} Gy$. Then f and G have a coincidence point in X.

Proof. Let $A = \bigcap_{n\in\mathbb{N}} (fG)^n X$. Lemma 1.1 ensures that A is a nonempty compact subset of X and $fGA = A$. Now we claim that for all $x \in X$,

$$f(fG)^n x \subseteq (fG)^n fx, \quad n \in \mathbb{N}; \tag{2.4}$$

$$G(fG)^n x \subseteq (fG)^n Gx, \quad n \in \mathbb{N}. \tag{2.5}$$

It follows from (2.1) that (2.4) holds for $n = 1$. Suppose that (2.4) holds for some $n \in \mathbb{N}$. It follows from (2.1) that

$$f(fG)^{n+1}x = \big[f(fG)^n\big]fGx \subseteq \big[(fG)^n f\big]fGx$$

$$= (fG)^n f(fGx) \subseteq (fG)^n fGfx = (fG)^{n+1} fx.$$

That is, (2.4) holds for $n+1$. By induction, (2.4) holds. Similarly, we can prove that (2.5) holds. (2.4) and (2.5) yield that

$$fA = f\bigcap_{n\in\mathbb{N}}(fG)^n X \subseteq \bigcap_{n\in\mathbb{N}} f(fG)^n X \subseteq \bigcap_{n\in\mathbb{N}}(fG)^n fX \subseteq A$$

and

$$GA = G\bigcap_{n\in\mathbb{N}}(fG)^n X \subseteq \bigcap_{n\in\mathbb{N}} G(fG)^n X \subseteq \bigcap_{n\in\mathbb{N}}(fG)^n GX \subseteq A.$$

Consequently, $A = fGA \subseteq fA$ and $A = fGA \subseteq GfA \subseteq GA$. Hence, $A = fA = GA$. From the continuity of f and G, it follows that $d(fz, Gz) = \inf\big\{d(fx, Gx) : x \in A)\big\}$ for some $z \in A$.

We claim that $fz \in Gz$. Otherwise, $fz\overline{\in}Gz$. There exists $y \in Gz$ such that $d(fz, Gz) = d(fz, y)$. Owing to $fA = A$, we can find $w \in A$ with $fw = y$ such that

$$0 < d(fz, y) = d(fz, fw) = d(fz, Gz) \le d(fw, Gw) \le H(Gz, Gw). \tag{2.6}$$

This means that $fz \ne fw$, $Gz \ne Gw$, $fz\overline{\in}Gz$ and $fw\overline{\in}Gw$. By virtue of (2.3) and (2.6), we derive that

$$\begin{aligned}
H(Gz, Gw) &= H(Gz, Gw) - \min\big\{d(fz, Gw), d(fw, Gz)\big\} \\
&< \max\Big\{d(fz, Gz), d(fw, Gw), d(fz, fw), D(fw, Gz) \\
&\quad \frac{1}{2}\big[d(fz, Gw) + d(fw, Gz)\big], \frac{1}{2}\big[H(Gz, Gw) + d(fz, Gz)\big], \\
&\quad \frac{1}{2}\big[H(Gz, Gw) + d(fw, Gw)\big], \frac{1}{2}\big[H(Gz, Gw) + d(fz, fw)\big], \\
&\quad \frac{d(fz, Gw)d(fw, Gz)}{H(Gz, Gw)}, \frac{d(fz, Gz)d(fw, Gw)}{d(fz, fw)}, \\
&\quad \frac{d(fz, Gw)d(fw, Gz)}{d(fz, fw)}, \frac{H(Gz, Gw)d(fz, Gz)}{d(fz, fw)}\Big\} \\
&= \max\Big\{d(fz, Gz), d(fw, Gw), d(fz, Gz), 0 \\
&\quad \frac{1}{2}d(fz, Gw), \frac{1}{2}\big[H(Gz, Gw) + d(fz, Gz)\big], \\
&\quad \frac{1}{2}\big[H(Gz, Gw) + d(fw, Gw)\big], \frac{1}{2}\big[H(Gz, Gw) + d(fz, Gz)\big], \\
&\quad 0, \frac{d(fz, Gz)d(fw, Gw)}{d(fz, Gz)}, 0, \frac{H(Gz, Gw)d(fz, Gz)}{d(fz, Gz)}\Big\}
\end{aligned}$$

$$\le \max\Big\{d(fw, Gw), H(Gz, Gw), \frac{1}{2}\big[H(Gz, Gw) + d(fw, Gw)\big]\Big\} = H(Gz, Gw),$$

which is a contradiction. This completes the proof. □

Theorem 2.2. *Let (X,d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let all powers of fG map X into $C(X)$ and f, G and $(fG)^m$ be continuous, where m is some element in $\mathbb{N}$. Suppose that f and G satisfy* (2.1), (2.2) *and*

$$\min\Big\{H(Gx,Gy),d(fx,Gx),d(fy,Gy),d(fx,fy),\frac{d(fx,Gx)d(fy,Gy)}{d(fx,fy)}\Big\}-\min\big\{d(fx,Gy),d(fy,Gx)\big\}$$
$$<\max\big\{d(fx,fy),\min\{d(fx,Gx),D(fy,Gy)\},\min\{d(fx,Gy),d(fy,Gx)\}\big\} \quad (2.7)$$

for all $x, y \in X$ with $fx \neq fy$, $fx\overline{\in}Gx$ and $fy\overline{\in}Gy$. Then f and G have a coincidence point in X.

Proof. Let $A = \bigcap_{n\in\mathbb{N}}(fG)^nX$. As in the proof of Theorem 2.1, we can prove that $fA = GA = A$. From the continuity of f and G, it follows that $d(fz, Gz) = \inf\big\{d(fx, Gx) : x \in A)\big\}$ for some $z \in A$.

We assert that $fz \in Gz$. Otherwise, $fz\overline{\in}Gz$. There exists $y \in Gz$ such that $d(fz, Gz) = d(fz, y)$. From $fA = A$, we can find $w \in A$ satisfying $fw = y$ and (2.6). Note that $fz \neq fw$, $fz\overline{\in}Gz$ and $fw\overline{\in}Gw$. In light of (2.6) and (2.7), we acquire that

$$\begin{aligned}
d(fz,Gz) &= \min\Big\{H(Gz,Gw),d(fz,Gz),d(fw,Gw),d(fz,Gz),\frac{d(fz,Gz)d(fw,Gw)}{d(fz,Gz)}\Big\}-\min\big\{d(fz,Gw),0\big\}\\
&= \min\Big\{H(Gz,Gw),d(fz,Gz),d(fw,Gw),d(fz,fw),\frac{d(fz,Gz)d(fw,Gw)}{d(fz,fw)}\Big\}-\min\big\{d(fz,Gw),d(fw,Gz)\big\}\\
&< \max\big\{d(fz,fw),\min\{d(fz,Gz),d(fw,Gw)\},\min\{d(fz,Gw),d(fw,Gz)\}\big\}\\
&= \max\big\{d(fz,Gz),\min\{d(fz,Gz),d(fw,Gw)\},\min\{d(fz,Gw),0\}\big\} = d(fz,Gz),
\end{aligned}$$

which is impossible. The proof is completed. □

Theorem 2.3. *Let (X,d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let all powers of fG map X into $C(X)$ and f, G and $(fG)^m$ be continuous, where m is some element in $\mathbb{N}$. Suppose that f and G satisfy* (2.1), (2.2) *and*

$$\min\big\{H^2(Gx,Gy),H(Gx,Gy)d(fx,Gx),$$

$$H(Gx,Gy)d(fy,Gy), H(Gx,Gy)d(fx,fy)\big\}$$
$$- \min\big\{d(fx,Gy)d(fx,fy), d(fy,Gx)d(fx,fy)\big\}$$
$$< \max\big\{H(Gx,Gy)d(fx,fy), d^2(fx,fy)\big\} \quad (2.8)$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx \overline{\in} Gx$ and $fy \overline{\in} Gy$. Then f and G have a coincidence point in X.

Proof. Let $A = \bigcap_{n\in\mathbb{N}}(fG)^n X$. As in the proof of Theorem 2.1, we can prove that $fA = GA = A$. From the continuity of f and G, it follows that $d(fz,Gz) = \inf\big\{d(fx,Gx) : x \in A)\big\}$ for some $z \in A$.

We now prove that $fz \in Gz$. Otherwise, $fz \overline{\in} Gz$. There exists $y \in Gz$ such that $d(fz,Gz) = d(fz,y)$. By $fA = A$, we can find $w \in A$ satisfying $fw = y$ and (2.6). Notice that $fz \neq fw$, $Gz \neq Gw$, $fz \overline{\in} Gz$ and $fw \overline{\in} Gw$. It follows from (2.6) and (2.8) that

$$\begin{aligned} H(Gz,Gw)d(fz,Gz) &= \min\big\{H^2(Gz,Gw), H(Gz,Gw)d(fz,Gz),\\ &\quad H(Gz,Gw)d(fw,Gw), H(Gz,Gw)d(fz,Gz)\big\}\\ &\quad - \min\big\{d(fz,Gw)d(fz,Gz), 0\big\}\\ &= \min\big\{H^2(Gz,Gw), H(Gz,Gw)d(fz,Gz),\\ &\quad H(Gz,Gw)d(fw,Gw), H(Gz,Gw)d(fz,fw)\big\}\\ &\quad - \min\big\{d(fz,Gw)d(fz,fw), d(fw,Gz)d(fz,fw)\big\}\\ &< \max\big\{H(Gz,Gw)d(fz,Gz), d^2(fz,Gz)\big\} = H(Gz,Gw)d(fz,Gz) \quad (2.1)\end{aligned}$$

which is a contradiction. This completes the proof. □

Theorem 2.4. *Let (X,d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let all powers of fG map X into $C(X)$ and f, G and $(fG)^m$ be continuous, where m is some element in $\mathbb{N}$. Suppose that f and G satisfy (2.1), (2.2) and*

$$\begin{aligned} H^2(Gx,Gy) &- \min\big\{d(fx,Gy)d(fx,fy), d(fy,Gx)d(fx,fy)\big\}\\ &< \max\big\{H(Gx,Gy)d(fx,Gx), H(Gx,Gy)d(fy,Gy),\\ &\quad H(Gx,Gy)d(fx,fy), d(fx,Gx)d(fx,fy),\\ &\quad d(fy,Gy)d(fx,fy), d(fx,Gx)d(fy,Gy), d(fx,Gy)d(fy,Gx)\big\} \quad (2.9)\end{aligned}$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx \overline{\in} Gx$ and $fy \overline{\in} Gy$. Then f and G have a coincidence point in X.

Proof. Let $A = \bigcap_{n\in\mathbb{N}}(fG)^n X$. As in the proof of Theorem 2.1, we can prove that $fA = GA = A$. From the continuity of f and G, it follows that $d(fz,Gz) = \inf\big\{d(fx,Gx) : x \in A)\big\}$ for some $z \in A$.

We claim that $fz \in Gz$. Otherwise, $fz \overline{\in} Gz$. There exists $y \in Gz$ such that $d(fz, Gz) = d(fz, y)$. In view of $fA = A$, we can find $w \in A$ satisfying $fw = y$ and (2.6). Clearly $fz \neq fw$, $Gz \neq Gw$, $fz \overline{\in} Gz$ and $fw \overline{\in} Gw$. From (2.6) and (2.9), we infer that

$$\begin{aligned}
H^2(Gz, Gw) &= H^2(Gz, Gw) - \min\big\{d(fz, Gw)d(fz, Gz), 0\big\} \\
&= H^2(Gz, Gw) - \min\big\{d(fz, Gw)d(fz, fw), d(fw, Gz)d(fz, fw)\big\} \\
&< \max\big\{H(Gz, Gw)d(fz, Gz), H(Gz, Gw)d(fw, Gw), \\
&\quad H(Gz, Gw)d(fz, fw), d(fz, Gz)d(fz, fw), \\
&\quad d(fw, Gw)d(fz, fw), d(fz, Gz)d(fw, Gw), \\
&\quad d(fz, Gw)d(fw, Gz)\big\} \\
&= \max\big\{H(Gz, Gw)d(fz, Gz), H(Gz, Gw)d(fw, Gw), \\
&\quad H(Gz, Gw)d(fz, Gz), d^2(fz, Gz), d(fw, Gw)d(fz, Gz), \\
&\quad d(fz, Gz)d(fw, Gw), 0\big\} = H(Gz, Gw)d(fw, Gw) \leq H^2(Gz, Gw),
\end{aligned}$$

which is impossible. The proof is completed. □

Theorem 2.5. *Let (X, d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let all powers of fG map X into $C(X)$ and f, G and $(fG)^m$ be continuous, where m is some element in $\mathbb{N}$. Suppose that f and G satisfy (2.1), (2.2) and*

$$\begin{aligned}
H^2(Gx, Gy) - \min\big\{d(fx, Gy)d(fx, fy), d(fy, Gx)d(fx, fy)\big\} \\
< \max\big\{d^2(fx, Gx), d^2(fy, Gy), \\
H(Gx, Gy)d(fx, fy), d(fx, Gx)d(fx, fy), \\
d(fy, Gy)d(fx, fy), d(fx, Gy)d(fy, Gx)\big\} \qquad (2.10)
\end{aligned}$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx \overline{\in} Gx$ and $fy \overline{\in} Gy$. Then f and G have a coincidence point in X.

Proof. Let $A = \bigcap_{n\in\mathbb{N}} (fG)^n X$. As in the proof of Theorem 2.1, we can prove that $fA = GA = A$. From the continuity of f and G, it follows that $d(fz, Gz) = \inf\big\{d(fx, Gx) : x \in A)\big\}$ for some $z \in A$.

We now show that $fz \in Gz$. Otherwise, $fz \overline{\in} Gz$. There exists $y \in Gz$ such that $d(fz, Gz) = d(fz, y)$. By virtue of $fA = A$, we can find $w \in A$ satisfying $fw = y$ and (2.6). Obviously $fz \neq fw, Gz \neq Gw, fz \overline{\in} Gz$ and $fw \overline{\in} Gw$. In view of (2.6) and (2.10), we derive that

$$\begin{aligned}
H^2(Gz, Gw) &= H^2(Gz, Gw) - \min\big\{d(fz, Gw)d(fz, Gz), 0\big\} \\
&= H^2(Gz, Gw) - \min\big\{d(fz, Gw)d(fz, fw), d(fw, Gz)d(fz, fw)\big\}
\end{aligned}$$

$$< \max\big\{d^2(fz,Gz), d^2(fw,Gw), H(Gz,Gw)d(fz,fw), d(fz,Gz)d(fz,fw), \\ d(fw,Gw)d(fz,fw), d(fz,Gw)d(fw,Gz)\big\} \\ = \max\big\{d^2(fz,Gz), d^2(fw,Gw), H(Gz,Gw)d(fz,Gz), \\ d(fz,Gz)d(fz,Gz), d(fw,Gw)d(fz,Gz), 0\big\} \le H^2(Gz,Gw),$$

which is a contradiction. This completes the proof. □

Theorem 2.6. *Let (X,d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let all powers of fG map X into $C(X)$ and f, G and $(fG)^m$ be continuous, where m is some element in $\mathbb{N}$. Suppose that f and G satisfy (2.1), (2.2) and*

$$\min\big\{H^2(Gx,Gy), H(Gx,Gy)d(fx,Gx), \\ H(Gx,Gy)d(fy,Gy), H(Gx,Gy)d(fx,fy), \\ d(fx,Gx)d(fx,fy), d(fy,Gy)d(fx,fy), \\ d(fx,Gx)d(fy,Gy)\big\} \\ - \min\big\{d(fx,Gy), d(fy,Gx)\big\}H(Gx,Gy) \\ < \max\big\{d^2(fx,fy), \min\{d(fx,Gy), d(fy,Gx)\}d(fx,fy)\big\} \tag{2.11}$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx\overline{\in}Gx$ and $fy\overline{\in}Gy$. Then f and G have a coincidence point in X.

Proof. Let $A = \bigcap_{n\in\mathbb{N}}(fG)^nX$. As in the proof of Theorem 2.1, we can prove that $fA = GA = A$. From the continuity of f and G, it follows that $d(fz,Gz) = \inf\big\{d(fx,Gx) : x \in A)\big\}$ for some $z \in A$.

We assert that $fz \in Gz$. Otherwise, $fz\overline{\in}Gz$. There exists $y \in Gz$ such that $d(fz,Gz) = d(fz,y)$. By $fA = A$, we can find $w \in A$ satisfying $fw = y$ and (2.6). Note that $fz \neq fw, Gz \neq Gw, fz\overline{\in}Gz$ and $fw\overline{\in}Gw$. By virtue of (2.6) and (2.11), we gain that

$$d^2(fz,Gz) = \min\big\{H^2(Gz,Gw), H(Gz,Gw)d(fz,Gz), H(Gz,Gw)d(fw,Gw), \\ H(Gz,Gw)d(fz,Gz), d(fz,Gz)d(fz,Gz), \\ d(fw,Gw)d(fz,Gz), d(fz,Gz)d(fw,Gw)\big\} \\ - \min\big\{d(fz,Gw), 0\big\}H(Gz,Gw) \\ = \min\big\{H^2(Gz,Gw), H(Gz,Gw)d(fz,Gz), H(Gz,Gw)d(fw,Gw), \\ H(Gz,Gw)d(fz,fw), d(fz,Gz)d(fz,fw), \\ d(fw,Gw)d(fz,fw), d(fz,Gz)d(fw,Gw)\big\} \\ - \min\big\{d(fz,Gw), d(fw,Gz)\big\}H(Gz,Gw)$$

$$< \max\left\{d^2(fz,fw), \min\{d(fz,Gw), d(fw,Gz)\}d(fz,fw)\right\}$$
$$< \max\left\{d^2(fz,Gz), \min\{d(fz,Gw), 0\}d(fz,fw)\right\} = d^2(fz,Gz),$$

which is impossible. This completes the proof. □

Theorem 2.7. *Let (X,d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let all powers of fG map X into $C(X)$ and f, G and $(fG)^m$ be continuous, where m is some element in $\mathbb{N}$. Suppose that f and G satisfy* (2.1), (2.2) *and*

$$\begin{aligned}\min&\left\{H^2(Gx,Gy), d^2(fx,Gx), d^2(fy,Gy),\right.\\ &\left.H(Gx,Gy)d(fx,fy), d(fx,Gx)d(fx,fy), d(fy,Gy)d(fx,fy)\right\}\\ &- \min\left\{H(Gx,Gy)d(fx,Gy), H(Gx,Gy)d(fy,Gx)\right\}\\ < \max&\left\{d^2(fx,fy), \min\{d(fx,Gy)d(fx,fy), d(fy,Gx)d(fx,fy)\}\right\}\end{aligned} \tag{2.12}$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx\overline{\in}Gx$ and $fy\overline{\in}Gy$. Then f and G have a coincidence point in X.

Proof. Let $A = \bigcap_{n\in\mathbb{N}}(fG)^n X$. As in the proof of Theorem 2.1, we can prove that $fA = GA = A$. From the continuity of f and G, it follows that $d(fz,Gz) = \inf\left\{d(fx,Gx) : x \in A)\right\}$ for some $z \in A$.

We now prove that $fz \in Gz$. Otherwise, $fz\overline{\in}Gz$. There exists $y \in Gz$ such that $d(fz,Gz) = d(fz,y)$. Owing to $fA = A$, we can find $w \in A$ satisfying $fw = y$ and (2.6). It follows that $fz \neq fw$, $Gz \neq Gw$, $fz\overline{\in}Gz$ and $fw\overline{\in}Gw$, (2.6) and (2.12) that

$$\begin{aligned}&d^2(fz,Gz)\\ &= \min\left\{H^2(Gz,Gw), d^2(fz,Gz), d^2(fw,Gw), H(Gz,Gw)d(fz,Gz),\right.\\ &\qquad\left. d(fz,Gz)d(fz,Gz), d(fw,Gw)d(fz,Gz)\right\}\\ &\quad - \min\left\{H(Gz,Gw)d(fz,Gw), 0\right\}\\ &= \min\left\{H^2(Gz,Gw), d^2(fz,Gz), d^2(fw,Gw), H(Gz,Gw)d(fz,fw),\right.\\ &\qquad\left. d(fz,Gz)d(fz,fw), d(fw,Gw)d(fz,fw)\right\}\\ &\quad - \min\left\{H(Gz,Gw)d(fz,Gw), H(Gz,Gw)d(fw,Gz)\right\}\\ &< \max\left\{d^2(fz,fw), \min\{d(fz,Gw)d(fz,fw), d(fw,Gz)d(fz,fw)\}\right\}\\ &< \max\left\{d^2(fz,Gz), \min\{d(fz,Gw)d(fz,Gz), 0\}\right\}\\ &= d^2(fz,Gz),\end{aligned}$$

which is impossible. The proof is completed. □

Theorem 2.8. *Let (X,d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let f and*

G be continuous, strongly commutative and satisfy (2.3). *Then f and G have a coincidence point in X.*

Proof. Let $A = \bigcap_{n\in\mathbb{N}} f^n X$. It follows from Lemma 1.1 that A is a nonempty compact subset of X and $fA = A$. Since f and G is strongly commutative, we infer that

$$Gf^n x = Gf(f^{n-1}x) \subseteq fGf^{n-1}x \subseteq \cdots \subseteq f^n Gx \tag{2.13}$$

for all $x \in X$ and $n \in \mathbb{N}$. In view of (2.13), we conclude that

$$GA = G\bigcap_{n\in\mathbb{N}} f^n X \subseteq \bigcap_{n\in\mathbb{N}} Gf^n X \subseteq \bigcap_{n\in\mathbb{N}} f^n GX \subseteq A.$$

The rest of the proof follows as in Theorem 2.1. This completes the proof. □

Similarly we have

Theorem 2.9. *Let (X, d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let f and G be continuous, strongly commutative and satisfy one of (2.7)-(2.12). Then f and G have a coincidence point in X.*

As a consequence of Theorem 2.1, we have

Corollary 2.1. *Let (X, d) be a compact metric space, $G : X \to C(X)$ be a multivalued mapping and $f : X \to X$ be a single valued mapping. Let all powers of fG map X into $C(X)$ and f, G, and $(fG)^m$ be continuous, where m is some element in $\mathbb{N}$. Suppose that f and G satisfy* (2.1), (2.2) *and the following*

$$\begin{aligned} &H(Gx, Gy) - \min\{d(fx, Gy), d(fy, Gx)\} \\ &< \max\Big\{d(fx, Gx), d(fy, Gy), d(fx, fy), \\ &\quad \frac{1}{2}\big[d(fx, Gy) + d(fy, Gx)\big], \frac{d(fx, Gy)d(fy, Gx)}{H(Gx, Gy)}, \\ &\quad \frac{d(fx, Gx)d(fy, Gy)}{d(fx, fy)}, \frac{d(fx, Gy)d(fy, Gx)}{d(fx, fy)}\Big\} \end{aligned} \tag{2.14}$$

for all $x, y \in X$ with $fx \neq fy$, $Gx \neq Gy$, $fx \overline{\in} Gx$ and $fy \overline{\in} Gy$. Then f and G have a coincidence point in X.

Remark 2.1. Corollary 2.1 extends Theorem 3.2 of Hu and Rosen [1], Theorem 2.2 of Liu [10] and Theorems 3 and 4 of Rao [11]. The following example shows that Corollary 2.1 does indeed generalize the corresponding results in [1], [10], [11].

Example 2.1. Let $X = \{1, 2, 5, 6\}$ with the usual metric. Let f be the identity mapping on X. Define a multivalued mapping $G : X \to C(X)$ by $G1 = \{2\}$, $G2 = \{1, 2\}$, $G5 = \{6\}$ and $G6 = \{5, 6\}$. Suppose that $x, y \in X$ with

$fx \neq fy$, $Gx \neq Gy$, $x\overline{\in}Gx$, $y\overline{\in}Gy$. Then $(x, y) \in \{(1,5),(5,1)\}$. It is easy to derive that

$$\begin{aligned} &H(Gx,Gy) - \min\big\{d(fx,Gy), d(fy,Gx)\big\} \\ &= 4 - 3 < 4 = d(fx,fy) \\ &= \max\Big\{d(fx,Gx), d(fy,Gy), d(fx,fy), \\ &\quad \frac{1}{2}\big[d(fx,Gy) + d(fy,Gx)\big], \frac{d(fx,Gy)d(fy,Gx)}{H(Gx,Gy)}, \\ &\quad \frac{d(fx,Gx)d(fy,Gy)}{d(fx,fy)}, \frac{d(fx,Gy)d(fy,Gx)}{d(fx,fy)}\Big\} \end{aligned}$$

for $(x, y) \in \{(1,5),(5,1)\}$. It is not difficult to validate that all the conditions of Corollary 2.1 are satisfied. However, Theorem 3.2 of Hu and Rosen [1], Theorem 2.2 of Liu [10] and Theorems 3 and 4 of Rao [11] are not applicable since $H(Gx, Gy) = 4 = d(x, y)$,

$$\begin{aligned} H(Gx,Gy) = 4 = \max\Big\{&d(fx,fy), d(fx,Gx), d(fy,Gy), \\ &\frac{1}{2}\big[d(fx,Gy) + d(fy,Gx)\big], \\ &\frac{d(fx,Gx)d(fy,Gy)}{d(fx,fy)}, \frac{d(fx,Gy)d(fy,Gx)}{d(fx,fy)}\Big\} \end{aligned}$$

and

$$\begin{aligned} H(Gx,Gy) = 4 = \max\Big\{&d(fx,fy), d(fx,Gx), d(fy,Gy), \\ &\frac{1}{2}\big[d(fx,Gy) + d(fy,Gx)\big]\Big\} \end{aligned}$$

for $(x, y) \in \{(1,5),(5,1)\}$.

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