International Journal of Pure and Applied Mathematics
Volume 53 No. 1 2009, 121-131

GENERALIZED DIFFERENCE OPERATOR OF THIRD KIND AND ON SECOND PARTIAL SUMS OF PRODUCTS OF CONSECUTIVE TERMS OF ARITHMETIC AND ARITHMETICO-GEOMETRIC PROGRESSIONS

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Abstract: In this paper, the authors extend the theory of the generalized difference operator Δ_ℓ and the second Δ_{ℓ_1,ℓ_2} to third kind $\Delta_{\ell_1,\ell_2,\ell_3}$ for the positive reals ℓ_1, ℓ_2 and ℓ_3 by presenting some results on generalized polynomial factorials of several kinds, generalized Leibnitz Theorem and Newton's formula. Also we develop a formula to find second partial sums of products of n consecutive terms of arithmetic, arithmetico-geometric progressions by defining the inverse operator $\Delta^{-1}_{\ell_1,\ell_2,\ell_3}$ of the third kind operator $\Delta_{\ell_1,\ell_2,\ell_3}$.

AMS Subject Classification: 39A
Key Words: generalized difference operator, generalized polynomial factorial, partial sums

1. Introduction

The basic theory of difference equations is based on the operator Δ defined as

Received: April 15, 2009

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$$\Delta u(k) = u(k+1) - u(k), \quad k \in \mathbb{N}, \tag{1}$$

where $\mathbb{N} = \{0, 1, 2, 3, \cdots, \}$. Eventhough many authors [1], [8]-[10] have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k), \qquad k \in \mathbb{N}, \qquad \ell \in \mathbb{R} - \{0\}, \tag{2}$$

no significant progress took place on this line. But recently, when we took up the definition of Δ as given in (2) and developed the theory of difference equations in a different direction, we obtained some interesting results in the applications of number theory. For convinence, we labelled the operator Δ defined by (2) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1} many interesting results on number theory were obtained (see [2]). By extending the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and weblike were studied for the solutions of difference equations involving Δ_ℓ. The results obtained can be found in [2]-[6]. The goal of this paper is to obtain some significant results on $\Delta_{\ell_1,\ell_2,\ell_3}$ and to obtain the formulae to find the values of C_n, PC_n, P^2C_n, where C_n denotes the sum of the products of n consecutive terms of an A.P., PC_n denotes the sum of all partial sums of C_n and P^2C_n (second partial sums of C_n) denotes the sum of all the partial sums of PC_n. The mathematical expressions of C_n, PC_n, P^2C_n are given below:

$$C_n = \prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell) + \prod_{r=2}^{n+1}(i+r\ell) + \cdots + \prod_{r=k}^{n+k-1}(i+r\ell),$$

$$\begin{aligned} PC_n = \prod_{r=0}^{n-1}(i+r\ell) + \left\{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell)\right\} + \cdots \\ + \left\{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell) + \cdots + \prod_{r=k}^{n+k-1}(i+r\ell)\right\}, \end{aligned}$$

$$\begin{aligned} P^2C_n = \prod_{r=0}^{n-1}(i+r\ell) + \left[\prod_{r=0}^{n-1}(i+r\ell) + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell)}\right] \\ + \left[\prod_{r=0}^{n-1}(i+r\ell) + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell)}\right] + \cdots \\ + \left[\prod_{r=0}^{n-1}(i+r\ell) + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell) + \cdots} \right. \end{aligned}$$

$$+\overline{\prod_{r=0}^{n-1}(i+r\ell)+\prod_{r=1}^{n}(i+r\ell)+\cdots+\prod_{r=k}^{n+k-1}(i+r\ell)}\Big] .$$

Let nS^n be the sums of an arithmetico-geometric progression, PnS^n be the sum of all the partial sums of nS^n and P^2nS^n be the sum of all the partial sums of PnS^n. Then they are expressed respectively as follows.

$$nS^n = ja^j + (j+\ell)a^{j+\ell} + (j+2\ell)a^{j+2\ell} + \cdots + (j+n\ell)a^{j+n\ell},$$

$$PnS^n = ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}} + \cdots + \overline{ja^j + \cdots + (j+n\ell)a^{j+n\ell}},$$

$$\begin{aligned} P^2nS^n = {} & ja^j + [ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}}] + [ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}} \\ & + \overline{ja^j + (j+\ell)a^{j+\ell} + (j+2\ell)a^{j+2\ell}}] + \cdots + [ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}} \\ & + \cdots + \overline{ja^j + (j+\ell)a^{j+\ell} + (j+2\ell)a^{j+2\ell} + \cdots + (j+n\ell)a^{j+n\ell}}]. \end{aligned}$$

The formulae for C_n, PC_n and nS^n, PnS^n are derived in [2], [7]. So, in this paper, we derive the formulae for finding the values of P^2C_n and P^2nS^n using $\Delta_{\ell,\ell,\ell}$ and present the derivations on the results involving $\Delta_{\ell_1,\ell_2,\ell_3}$ and the reader can find the results involving Δ_{ℓ_1,ℓ_2} in [2], [7].

Throughout this paper, we make use of the following assumptions:

(i) $N = \{0, 1, 2, 3, ...\}$.

(ii) $N(a) = \{a, a+1, a+2, ...\}$.

(iii) r and n are positive integers and ℓ_1, ℓ_2 and ℓ_3 are positive reals.

(iv) n^* is the largest non negative integer such that $k - n^*\ell \geq 0$.

(v) $c, c_0, c_1, c_2, \ldots$ are constants.

(vi) $rC_i = \frac{r!}{(r-i)!i!}$ where $0! = 1$, $r! = 1.2.3...r$.

(vii) $u : [0,\infty) \to \mathbb{C}$ is a complex valued function on $[0,\infty)$.

2. Preliminaries

In this section, we present some preliminary definitions and results which will be useful for deriving our main results.

Definition 2.1. Let $u : [0,\infty) \to \mathbb{C}$ be any complex valued function on $[0,\infty)$. We define the generalized difference operator of the third kind for $u(k)$ as

$$\Delta_{\ell_1,\ell_2,\ell_3}u(k) = u(k+\ell_1+\ell_2+\ell_3) - [u(k+\ell_1+\ell_2) + u(k+\ell_1+\ell_3)$$

$$+ u(k+\ell_2+\ell_3)] + [u(k+\ell_1) + u(k+\ell_2) + u(k+\ell_3)] - u(k). \quad (3)$$

Lemma 2.2. *If E is the usual shift operator, then the following are simple to derive. If $\ell_j, j = 1, 2, 3$ are positive reals, then*

$$(i)\ \Delta_{\ell_1,\ell_2,\ell_3} = E^{\ell_1+\ell_2+\ell_3} - (E^{\ell_1+\ell_2} + E^{\ell_1+\ell_3} + E^{\ell_2+\ell_3}) + (E^{\ell_1} + E^{\ell_2} + E^{\ell_3}) - I, \quad (4)$$

$$(ii)\ \Delta_{\ell_1,\ell_2,\ell_3} = \Delta_{\ell_1+\ell_2+\ell_3} - (\Delta_{\ell_1+\ell_2} + \Delta_{\ell_1+\ell_3} + \Delta_{\ell_2+\ell_3}) + (\Delta_{\ell_1} + \Delta_{\ell_2} + \Delta_{\ell_3}), \quad (5)$$

$$(iii)\ \Delta_{\ell_1,\ell_2,\ell_3} = \Delta_{\ell_1}\ \Delta_{\ell_2}\Delta_{\ell_3}, \quad (6)$$

$$(iv)\ \Delta_{\ell_1,\ell_2,\ell_3} = \prod_{j=1}^{3}\left(\sum_{i=1}^{\ell_j} \ell_j C_i \Delta^i\right). \quad (7)$$

Lemma 2.3. *If $m = 3n$, then $\Delta^n_{\ell,\ell,\ell} k^m = (3n)!\ \ \ell^{3n}$.*

Remark 2.4. If $P_k = a_0 k^{3n} + a_1 k^{3n-1} + \cdots + a_n$, then $\Delta^n_{\ell,\ell,\ell} P_k = a_0\ (3n)!\ \ \ell^{3n}$.

Definition 2.5. If n is a positive integer, then the generalized polynomial factorial in k of the third kind denoted by $k^{(n)}_{\ell_1,\ell_2,\ell_3}$ is defined as

$$k^{(n)}_{\ell_1,\ell_2,\ell_3} = (k+\ell_2+\ell_3)^{(n)}_{\ell_1} + (k+\ell_1+\ell_3)^{(n)}_{\ell_2} + (k+\ell_1+\ell_2)^{(n)}_{\ell_3}$$
$$- \{(k+\ell_2)^{(n)}_{\ell_1} + (k+\ell_3)^{(n)}_{\ell_1} + (k+\ell_1)^{(n)}_{\ell_2} + (k+\ell_3)^{(n)}_{\ell_2} + (k+\ell_1)^{(n)}_{\ell_3} + (k+\ell_2)^{(n)}_{\ell_3}\}$$
$$+ k^{(n)}_{\ell_1} + k^{(n)}_{\ell_2} + k^{(n)}_{\ell_3}. \quad (8)$$

Definition 2.6. The inverse of the generalized difference operator of the third kind denoted by $\Delta^{-1}_{\ell_1,\ell_2,\ell_3}$ is defined as follows. If $\Delta_{\ell_1,\ell_2,\ell_3} z(k) = y(k)$, then

$$z(k) = \Delta^{-1}_{\ell_1,\ell_2,\ell_3} y(k) + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k^{(1)}_{\ell_1}}{\ell_1}\right) + c_{0j}. \quad (9)$$

Lemma 2.7. *If s^n_r's are the Stirling numbers of the first kind, then*

$$k^{(n)}_{\ell,\ell,\ell} = 3\left\{\sum_{r=1}^{n} s^n_r \ell^{n-r} \Delta_{\ell,\ell} k^r\right\}.$$

Proof. The proof follows from the relation $\sum_{r=1}^{n} s^n_r \ell^{n-r} k^r = k^{(n)}_{\ell}$. □

Theorem 2.8. *There exits constants c_{0j}, c_{1j} and c_{2j} depend on $k - n^*\ell$*

such that

$$\Delta^{-1}_{\ell,\ell,\ell}y(k) = \sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*} y(k-t\ell-s\ell-r\ell) + c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}, \quad (10)$$

where $j = k - n^*\ell$ *and*

$$\Delta^{-1}_{\ell,\ell,\ell}k_\ell^{(n)} = \frac{k_\ell^{(n+3)}}{(n+1)(n+2)(n+3)\ \ell^3} + c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}. \quad (11)$$

Proof. The proof follows from the relation $\Delta_{\ell,\ell,\ell}\{\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*} y(k-t\ell-s\ell-r\ell) + c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}\} = y(k)$. □

3. Main Results

In this section, we present the generalized discrete version of the Leibnitz theorem, Newton's formula with respect to $\Delta_{\ell,\ell,\ell}$.

Theorem 3.1. *If* $u : [0,\infty) \to \mathbb{C}$ *and* $v : [0,\infty) \to \mathbb{C}$ *are any two functions, then*

$$\Delta^n_{\ell,\ell,\ell}(u(k)v(k)) = \Delta^n_{\ell,\ell}(u(k)\Delta^n_\ell v(k)) + nC_1\Delta^n_{\ell,\ell}(\Delta_\ell u(k)\Delta_\ell^{n-1}v(k+\ell)) + nC_2\Delta^n_\ell(\Delta^2_\ell u(k)\Delta_\ell^{n-2}v(k+2\ell)) + \cdots + nC_n\Delta^n_{\ell,\ell}(\Delta^n_\ell u(k)v(k+n\ell)).$$

Proof. The proof follows by the generalized Leibnitz Theorem [2] and (6). □

Using Stirling numbers of the first kind s^n_r, the following can be easily obtained.

Lemma 3.2. *If* ℓ, m *and* n *are positive integers then,* $\Delta_{\ell,\ell,\ell}k^n_{\ell,\ell,\ell} = (n\ \ell)^{(3)}_\ell\, k^{(n-3)}_{\ell,\ell,\ell}$ *and hence*

$$\Delta^m_{\ell,\ell,\ell}k^n_{\ell,\ell,\ell} = \begin{cases} n!\,3\ell^n, & \text{if} \quad n = 3m+2\,, \\ 0, & \text{if} \quad n < 3m+2\,. \end{cases}$$

Proof. The proof follows by (6) and induction on m. □

The following theorem is the generalized version of Newton's formula with reference to $\Delta_{\ell,\ell,\ell}$ and $k^{(n)}_{\ell,\ell,\ell}$.

Theorem 3.3. *Let* $f(k)$ *be a polynomial of degree* $(3n+2)$ *in* k. *Then*

$f(k)$ can be expressed as

$$f(0)+\frac{\Delta_{\ell,\ell,\ell}f(0)}{5!3\ell^5}k^{(5)}_{\ell,\ell,\ell}+\frac{\Delta^2_{\ell,\ell,\ell}f(0)}{8!3\ell^8}k^{(8)}_{\ell,\ell,\ell}+\cdots+\frac{\Delta^n_{\ell,\ell,\ell}f(0)}{(3n+2)!3\ell^{(3n+2)}}k^{(3n+2)}_{\ell,\ell,\ell}. \quad (12)$$

Proof. Assume that

$$f(k)=a_0+a_1k^{(5)}_{\ell,\ell,\ell}+a_2k^{(8)}_{\ell,\ell,\ell}+\cdots+a_nk^{(3n+2)}_{\ell,\ell,\ell}. \quad (13)$$

Clearly $f(0)=a_0$. The coefficients are determined from the relation

$$\Delta^r_{\ell,\ell,\ell}f(0)=(3r+2)!\ \ 3\ell^{3r+2}\ \ a_r, \quad r>0. \quad (14)$$

The proof follows from (13) and (14). □

Corollary 3.4. *Let $f(k)$ be a polynomial of degree $(3n+2)$ in k. Then $f(k-t)$ can be expressed as*

$$f(k-t)=f(t)+\frac{\Delta_{\ell,\ell,\ell}f(t)}{5!3\ell^5}(k-t)^{(5)}_{\ell,\ell,\ell}+\frac{\Delta^2_{\ell,\ell,\ell}f(t)}{8!3\ell^8}(k-t)^{(8)}_{\ell,\ell,\ell}+\cdots$$
$$+\frac{\Delta^n_{\ell,\ell,\ell}f(t)}{(3n+2)!3\ell^{(3n+2)}}(k-t)^{(3n+2)}_{\ell,\ell,\ell}.$$

Proof. The proof follows by replacing k by $(k-t)$ and 0 by t in (12). □

Lemma 3.5. *There exists constants $c_{\ell_1}, c_{\ell_2}, c_{\ell_3}$ and c such that*

$$\Delta^{-1}_{\ell_1,\ell_2,\ell_3}\big(k^{(n)}_{\ell_1,\ell_2,\ell_3}\big)=\frac{k^{(n+1)}_{\ell 1}}{\ell_1(n+1)}+\frac{k^{(n+1)}_{\ell 2}}{\ell_2(n+1)}+\frac{k^{(n+1)}_{\ell 3}}{\ell_3(n+1)}+c_{\ell_1}+c_{\ell_2}+c_{\ell_3}$$

and hence

$$\Delta^{-1}_{\ell,\ell,\ell}\big(k^{(n)}_{\ell,\ell,\ell}\big)=3\ \frac{k^{(n+1)}_{\ell}}{\ell(n+1)}+c.$$

Proof. The proof follows from (8) and the relation $\Delta_{\ell,\ell,\ell}\left(\frac{3\ k^{(n+1)}_{\ell}}{\ell(n+1)}+c\right)=k^{(n)}_{\ell,\ell,\ell}$. □

4. Applications

The following theorem gives a general rule to find the sum of the second partial sums of the products of n consecutive terms of an arithmetic progression.

Theorem 4.1. *If $i \geq (n-1)\ell$, then*

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}(k-t\ell-s\ell-r\ell)_\ell^{(n)}$$
$$=\frac{1}{(n+1)(n+2)(n+3)\ell^3}[k_\ell^{(n+3)}-((n+2)\ell+i)_\ell^{(n+3)}]$$
$$+\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}((n+2)\ell-t\ell-s\ell-r\ell+i)_\ell^{(n)}+\left((n+2)\ell+\frac{i-k}{\ell}\right)$$
$$\Big\{\frac{1}{(n+1)(n+2)(n+3)\ell^3}\left[((n+3)\ell+i)_\ell^{(n+3)}-((n+2)\ell+i)_\ell^{(n+3)}\right]$$
$$-\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}((n+3)\ell-t\ell-s\ell+i)_\ell^{(n)}\Big\}+\big[\frac{1}{2\ \ell^2}[((n+2)\ell+i)(2k-((n+3)\ell+i))$$
$$-k_\ell^{(2)}]]\Big\{\frac{1}{(n+1)(n+2)(n+3)\ell^3}\Big[((n+4)\ell+i)_\ell^{(n+3)}-$$
$$((n+2)\ell+i)_\ell^{(n+3)}\Big]+(-2)\Big(\frac{1}{(n+1)(n+2)(n+3)\ell^3}\Big[((n+3)\ell+i)_\ell^{(n+3)}$$
$$-((n+2)\ell+i)_\ell^{(n+3)}\Big]\Big)-\sum_{t=2}^{n^*}((n+3)\ell-t\ell+i)_\ell^{(n)}\Big\}. \quad (15)$$

Proof. From (10) and (11), we obtain

$$\Delta_{\ell,\ell,\ell}^{-1}K_\ell^{(n)}=\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}(k-t\ell-s\ell-r\ell)_\ell^{(n)}+c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right)+c_{1j}\left(\frac{k}{\ell}\right)+c_{0j}$$
$$=\frac{k_\ell^{(n+3)}}{(n+1)(n+2)(n+3)\ell^3}. \quad (16)$$

Put $k=((n+2)\ell+i)$ in (16) we obtain

$$c_{0j}=\frac{((n+2)\ell+i)_\ell^{(n+3)}}{(n+1)(n+2)(n+3)\ell^3}-\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}((n+2)\ell-t\ell-s\ell-r\ell+i)_\ell^{(n)}$$
$$-c_{2j}\left(\frac{((n+2)\ell+i)_\ell^{(2)}}{2\ell^2}\right)-c_{1j}\left(\frac{((n+2)\ell+i)_\ell}{\ell}\right). \quad (17)$$

Substituting (17) in (16) and replacing k by $((n+3)\ell+i)$, we obtain

$$c_{1j}=\frac{1}{(n+1)(n+2)(n+3)\ell^3}-\Big[((n+3)\ell+i)_\ell^{(n+3)}-$$
$$((n+2)\ell+i)_\ell^{(n+3)}\Big]+\frac{c_{2j}}{2\ell^2}\Big[((n+2)\ell+i)_\ell^{(2)}-((n+3)\ell+i)_\ell^{(2)}\Big]$$

$$-\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}((n+3)\ell+i-t\ell-s\ell)_\ell^{(n)}. \quad (18)$$

Substituting (17) and (18)in (16) and replacing k by $((n+4)\ell+i)$, we obtain

$$c_{2j} = \frac{1}{(n+1)(n+2)(n+3)\ell^3}\Big[((n+4)\ell+i)_\ell^{(n+3)} -$$
$$((n+2)\ell+i)_\ell^{(n+3)}\Big] + \sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}((n+2)\ell+i-t\ell-s\ell-r\ell)_\ell^{(n)}$$
$$+(-2)\Big\{\frac{1}{(n+1)(n+2)(n+3)\ell^3}\Big[((n+3)\ell+i)_\ell^{(n+43}$$
$$-((n+2)\ell+i)_\ell^{(n+3)}\Big] - \sum_{t=2}^{n^*}\sum_{s=1}^{n^*}((n+3)\ell+i-t\ell-s\ell)_\ell^{(n)}\Big\}$$
$$-\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}((n+4)\ell+i-t\ell-s\ell-r\ell)_\ell^{(n)}. \quad (19)$$

The proof now follows from (16), (17), (18) and (19). □

The following example is the illustration of Theorem (4.1).

Example 4.2. For the A.P., $3, 9, 15, 21, \cdots$, we find C_n, PC_n and P^2C_n for $n = 5$.

Solution. Since $n = 5, \ell = 6, i = 27$ and $k = 99$ we are interested in the evaluation of

$$P^2C_n = (3)(9)(15)(21)(27) + [(3)(9)(15)(21)(27)$$
$$+\overline{(3)(9)(15)(21)(27) + (9)(15)(21)(27)(33)}] + \cdots$$
$$+[(3)(9)(15)(21)(27) + \overline{(3)(9)(15)(21)(27) + (9)(15)(21)(27)(33)} + \cdots$$
$$+\overline{(3)(9)(15)(21)(27) + (9)(15)(21)(27)(33) + \cdots + (57)(63)(69)(75)(81)}]$$
$$P^2C_n = 16341843560.$$

Similarly, we can evaluate the value of $PC_n = 8055125595$ and $C_n = 3637724580$ using Δ_ℓ, $\Delta_{\ell,\ell}$ (see [2], [7]).

Lemma 4.3. *If a is nonzero, then*

$$\Delta^{-1}_{\ell,\ell,\ell} ka^k = \frac{a^k}{(a^\ell-1)^3}\left[k - \frac{3\ell a^\ell}{(a^\ell-1)}\right] + \frac{c_{2j}}{2\ell^2}k_\ell^{(2)} + \frac{c_{1j}}{\ell}k + c_{0j}. \quad (20)$$

Proof. The proof follows by the definition of

$$\Delta_{\ell,\ell,\ell}\Big\{\frac{a^k}{(a^\ell-1)^3}\left[k-\frac{3\ell a^\ell}{(a^\ell-1)}\right]+\frac{c_{2j}}{2\ell^2}k_\ell^{(2)}+\frac{c_{1j}}{\ell}k+c_{0j}\Big\}=ka^k. \qquad \square$$

The following theorem is to find the formula for sum of the sums of partial sums of an arithmetico-geometric progression.

Theorem 4.4. *If $k>3\ell$ and $k-n^*\ell>0$, then*

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}(k-t\ell-s\ell-r\ell)\ a^{(k-t\ell-s\ell-r\ell)}$$

$$=\frac{1}{(a^\ell-1)^3}[k\ a^k-(\ell+i)a^{\ell+i}]-\frac{3\ell\ a^\ell}{(a^\ell-1)^4}[a^k-a^{(\ell+i)}]+\left(1+\frac{i-k}{\ell}\right)$$
$$\left\{\frac{1}{(a^\ell-1)^3}[(2\ell+i)\ a^{(2\ell+i)}-(\ell+i)a^{\ell+i}]-\frac{3\ell\ a^\ell}{(a^\ell-1)^4}[a^{(2\ell+i)}-a^{(\ell+i)}]\right\}$$
$$+\left(\frac{(\ell+i)(2k-(2\ell+i))-k_\ell^{(2)}}{2\ \ell^2}\right)\Big\{\frac{1}{(a^\ell-1)^3}[(3\ell+i)\ a^{(3\ell+i)}-(\ell+i)a^{\ell+i}]$$
$$-\frac{3\ \ell\ a^\ell}{(a^\ell-1)^4}[a^{(3\ell+i)}-a^{(\ell+i)}]+(-2)\Big(\frac{1}{(a^\ell-1)^3}[(2\ell+i)\ a^{(2\ell+i)}-(\ell+i)a^{\ell+i}]$$
$$-\frac{3\ \ell\ a^\ell}{(a^\ell-1)^4}[a^{(2\ell+i)}-a^{(\ell+i)}]\Big)-ia^i\Big\}. \tag{21}$$

Proof. From (10) and (20), we obtain

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}(k-t\ell-s\ell-r\ell)\ a^{(k-t\ell-s\ell-r\ell)}+\frac{c_{2j}}{2\ell^2}k_\ell^{(2)}+\frac{c_{1j}}{\ell}k+c_{0j}$$
$$=\frac{a^k}{(a^\ell-1)^3}\left[k-\frac{3\ \ell\ a^\ell}{(a^\ell-1)}\right]. \tag{22}$$

Putting $k=(\ell+i)$ in (22) we obtain

$$c_{0j}=\frac{a^{(\ell+i)}}{(a^\ell-1)^3}\left[(\ell+i)-\frac{3\ \ell\ a^\ell}{(a^\ell-1)}\right]-\frac{c_{2j}}{2\ell^2}(\ell+i)_\ell^{(2)}-\frac{c_{1j}}{\ell}(\ell+i). \tag{23}$$

Substituting (23) in (22) and replacing k by $(2\ell+i)$, we get

$$c_{1j}=\frac{1}{(a^\ell-1)^3}\left[(2\ell+i)a^{(2\ell+i)}-(\ell+i)a^{(\ell+i)}\right]-\frac{3\ \ell\ a^\ell}{(a^\ell-1)^4}\Big[a^{(2\ell+i)}$$
$$-a^{(\ell+i)}\Big]+\frac{c_{2j}}{2\ell^2}\left[(\ell+i)_\ell^{(2)}-(2\ell+i)_\ell^{(2)}\right]. \tag{24}$$

Substituting (24) and (23) in (22) and replacing k by $(3\ell+i)$, we obtain

$$c_{2j} = \frac{1}{(a^{\ell}-1)^3}\left[(3\ell+i)a^{(3\ell+i)} - (\ell+i)a^{(\ell+i)}\right] - \frac{3\ \ell\ a^{\ell}}{(a^{\ell}-1)^4}\left[a^{(3\ell+i)} - a^{(\ell+i)}\right] + (-2)\left\{\frac{1}{(a^{\ell}-1)^3}\left[(2\ell+i)a^{(2\ell+i)} - (\ell+i)a^{(\ell+i)}\right] - \frac{3\ \ell\ a^{\ell}}{(a^{\ell}-1)^4}\left[a^{(2\ell+i)} - a^{(\ell+i)}\right]\right\} - ia^{i}\,. \quad (25)$$

The proof now follows from (22), (23), (24) and (25). □

The following example is the illustration of Theorem 4.4.

Example 4.5. For the arithmetico-geometric progression, $(3)4^3$, $(8)4^8$, $(13)4^{13}$, $\cdots$, $(48)4^{48}$, find nS^n, PnS^n, P^2nS^n.

Solution. Taking $i=3, a=4, \ell=5, k=63$ in (21), we have to evaluate

$$P^2nS^n = (3)4^3 + [(3)4^3 + \overline{(3)4^3 + (8)4^8}] + \cdots + [(3)4^3 + \overline{(3)4^3 + (8)4^8} + \cdots + \overline{(3)4^3 + (8)4^8 + \cdots + (48)4^{48}}].$$

$$P^2nS^n = 3.812949944 \times 10^{30}.$$

Similarly, we can evaluate the value of $PnS^n = 3.809614351 \times 10^{30}$ and $nS^n = 3.806281638 \times 10^{30}$ using Δ_{ℓ}, $\Delta_{\ell,\ell}$ (see [2], [7]).

Acknowledgements

This research is supported by University Grants Commission, New Delhi.

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