International Journal of Pure and Applied Mathematics
Volume 53 No. 2 2009, 153-162

ON COFINITELY RAD-SUPPLEMENTED MODULES

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Abstract: Let R be a ring and M be a left $R-$module. In this work some properties of (amply) cofinitely Rad-supplemented modules are developed. It is shown that if M contains a nonzero semi-hollow submodule then M is cofinitely Rad-supplemented if and only if M/N is cofinitely Rad-supplemented. Morever a module M with small radical is cofinitely Rad-supplemented such that Rad-supplements are supplements in M, then M is cofinitely supplemented. In addition, a ring R is left Rad-supplemented if and only if every left R-module is amply cofinitely Rad-supplemented. Also, we give a characterization of generalized semiperfect modules.

AMS Subject Classification: 16D10, 16D99
Key Words: cofinite submodule, Rad-supplement, (amply) cofinitely Rad-supplemented module, generalized projective cover

1. Introduction

In this note R will be an associative ring with identity and all modules unital left $R-$modules. Let M be an $R-$module. The notation $N \leq M$ means that N is a submodule of M. A submodule K of an $R-$module M is called *small* in M (notation $K \ll M$) if $K + L \neq M$ for every submodule L of M. An

Received: February 27, 2009

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epimorphism $f : K \to M$ is called a *small cover* if $\ker f \ll K$. A module M is *supplemented* if every submodule U of M has a *supplement* (in M), i.e. a submodule V minimal with respect to $U + V = M$. A submodule V of M is supplement of U in M if and only if $U + V = M$ and $U \cap V \ll V$ [7].

Rad$M$ will indicate Jacobson radical of M. A module M is called *semi-hollow* if every finitely generated proper submodule is small in M, or equality Rad$M = M$, see [5].

Let M be a module and U, V be submodules of M. A submodule V of M is called *radical supplement* or briefly Rad-*supplement* (according to [6], generalized supplement) of U in M if $U + V = M$ and $U \cap V \subseteq \text{Rad}V$ (see [5], Theorem 10.14). A module M is called Rad-*supplemented* (according to [6], generalized supplemented module) if every submodule U of M has a Rad-supplement in M. A submodule U of M has *ample* Rad-*supplements in* M if every submodule V of M such that $M = U + V$ contains a Rad-supplement of U in M. The module M is called *amply* Rad-*supplemented* (GAS-module in [6]) if every submodule has ample Rad-supplements in M. It is clear that every amply Rad-supplemented module is Rad-supplemented, but a Rad-supplemented module need not to be amply Rad-supplemented (see [6], Example 2.21).

Since Jacobson radical of a module M is the sum of all small submodules of M, every supplement is a Rad-supplement. Then, clearly every supplemented module is Rad-supplemented but a Rad-supplemented module need not to be supplemented. Note that semi-hollow modules are Rad-supplemented. Let R be a non-local dedekind domain with quotient field K. Then K is Rad-supplemented, but it is not supplemented.

Recall from [2] that a submodule N of M is called *cofinite* if the factor module M/N is finitely generated. In [4], a module M is called *cofinitely* Rad-*supplemented* if every cofinite submodule has a Rad-supplement in M and the closure properties of cofinitely Rad-supplemented modules is given.

In this paper, we obtain some properties of cofinitely Rad-supplemented modules in Section 2. In Section 3 we introduce amply cofinitely Rad-supplemented modules. We study the properties of amply cofinitely Rad-supplemented modules. Also, we show that a finitely generated module M is generalized semiperfect if and only if every maximal submodule of M has ample Rad-supplements which have a generalized projective cover.

2. Cofinitely Rad-Supplemented Modules

Clearly every Rad-supplemented module is cofinitely Rad-supplemented. It is shown [4] that a cofinitely Rad-supplemented module need not to be Rad-supplemented. We also give an example of such modules. Firstly we need following lemma.

Lemma 2.1. (see [1], Lemma 2) *Let R be a dedekind domain with quotient field K and $\{P_i\}_{i\in I}$ be an infinite collection of distinct maximal ideals of R. Let $M=\prod_{i\in I}(R/P_i)$ be the direct product of the simple R-modules R/P_i and $T = T(M)$ be the torsion submodule of M. Then M/T is divisible, therefore $M/T \cong K^J$ for some index set J and* $\mathrm{Rad}M = 0$.

Example 2.2. Let $R = \mathbb{Z}$, M be as in Lemma 2.1 and let $T = \bigoplus_{i\in I}(\mathbb{Z}/P_i)$ be the torsion submodule of M. By Lemma 2.1, since $\mathbb{Q}$ is the field of quotients of $\mathbb{Z}$, there is a submodule N of M such that $N/T \cong \mathbb{Q}$. Since $\mathbb{Q}$ does not contain a maximal submodule, N does not contain a maximal submodule containing T. Now we will show that N is a cofinitely Rad-supplemented, but it is not Rad-supplemented. Let U be any cofinite submodule of N. Then there is a maximal submodule P of N such that P contains U and it follows that $P + T = N$. Note that T is semi-simple. Then there is a submodule X of T such that $(P \cap T) \oplus X = T$ and therefore $P \cap X = 0$. It follows that

$$N = P + T = P + (P \cap T) \oplus X = P + X = P \oplus X.$$

Hence by Lemma 2.7 in [2], U is a direct summand of N and so N is cofinitely Rad-supplemented.

Suppose that N is a Rad-supplemented module. Then, every submodule of N is a direct summand of N since $\mathrm{Rad}N = 0$. This is a contradiction because $\mathbb{Q}$ is not semi-simple. Hence N is cofinitely Rad-supplemented but it is not Rad-supplemented.

We now begin by showing some properties of cofinitely Rad-supplemented modules.

Lemma 2.3. *Let M be an R-module and V be a* Rad-*supplement of U in M. Then $(V + L)/L$ is a* Rad-*supplement of U/L in M/L for every submodule L of U.*

Proof. If $M = U + V$, then $M = U + (V + L)$. Therefore $M/L = U/L + (V + L)/L$ for every submodule L of U. Since V is a Rad-supplement of U in M, $U \cap V \subseteq \mathrm{Rad}V$ and it follows that $U \cap V \subseteq \mathrm{Rad}(V + L)$. Hence, we get;

$$(U/L) \cap ((V + L)/L) = (U \cap V + L)/L.$$

Let $p : (V + L) \to (V + L)/L$ be the canonical epimorphism. By Theorem 2.8(1) in [5],

$$p(U \cap V) \subseteq p(\mathrm{Rad}(V + L) \subseteq \mathrm{Rad}((V + L)/L).$$

Since $p(U \cap V) = (U \cap V + L)/L$, we have $(U/L) \cap ((V + L)/L) \subseteq \mathrm{Rad}((V + L)/L)$. This completes the proof. □

Proposition 2.4. *Let M be a cofinitely* Rad*-supplemented module and N be a submodule with* $\mathrm{Rad}M \subseteq N$*. If* $\mathrm{Rad}(M/N) = \{N\}$*, every cofinite submodule of M/N is a direct summand of M/N.*

Proof. Let U/N be any cofinite submodule of M/N. Note that

$$(M/N)/(U/N) \cong M/U.$$

Then U is a cofinite submodule of M. By the hypothesis, there is a submodule V of M such that $U + V = M$ and $U \cap V \subseteq \mathrm{Rad}V$. By Lemma 2.3, $(V + N)/N$ is a Rad-supplement of U/N in M/N. Since $\mathrm{Rad}M \subseteq N$, $((V + N)/N) \cap (U/N) = \{N\}$ and it follows that U/N is a direct summand of M/N. □

Corollary 2.5. *Let M be a cofinitely* Rad*-supplemented module, then every cofinite submodule of $M/\mathrm{Rad}M$ is a direct summand of $M/\mathrm{Rad}M$.*

The $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is cofinitely Rad-supplemented, but $\mathbb{Z}$-module $\mathbb{Z}$ is not (cofinitely) Rad-supplemented. This example shows that the factor module of a module M is cofinitely Rad-supplemented but M need not to be cofinitely Rad-supplemented. We can prove the following theorem. □

Theorem 2.6. *Let M be an R-module and let N be a nonzero semi-hollow submodule of M. Then, M is cofinitely* Rad*-supplemented if and only if M/N is cofinitely* Rad*-supplemented.*

Proof. Let M be a cofinitely Rad-supplemented module. Consider U/N as a cofinite submodule of M/N. Then U is cofinite. Since M is cofinitely Rad-supplemented, there is a submodule V of M such that $U + V = M$ with $U \cap V \subseteq \mathrm{Rad}V$. By Lemma 2.3, $(V + N)/N$ is a Rad-supplement of U/N in M/N. Hence M/N is cofinitely Rad-supplemented. Conversely, suppose that U be a cofinite submodule of M. Then $(U + N)/N$ is a cofinite submodule of M/N. Since M/N is cofinitely Rad-supplemented, $(U + N)/N$ has a Rad-supplement in M/N. Suppose that V/N is Rad-supplement of $(U + N)/N$ in M/N. Then $U + V = M$. Since N is semi-hollow, $\mathrm{Rad}N = N$ and it follows that $U \cap V \subseteq \mathrm{Rad}V$. Hence M is cofinitely Rad-supplemented. □

Let M be an R-module. The module M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M, see [2]. Every cofinitely supplemented module is cofinitely Rad-supplemented, but it is not generally true that every cofinitely Rad-supplemented module is cofinitely supplemented. For example, a left Rad-supplemented ring which is not semiperfect is a cofinitely Rad-supplemented over itself, but is not cofinitely supplemented, see [4].

We give an analogue of such modules. Firstly we need the following lemma.

Lemma 2.7. *Let M be an R-module with small Jacobson radical and $U \leq M$.If U has a* Rad*-supplement that is a supplement in M, then U has a supplement in M.*

Proof. Let V be a Rad-supplement of U in M. Note that

$$U \cap V \subseteq \mathrm{Rad} V \subseteq \mathrm{Rad} M \ll M$$

and so $U \cap V \ll M$. Since V is a supplement in M, $U \cap V \ll V$. Hence V is a Rad-supplement of U in M. □

Theorem 2.8. *Let M be an R-module with small radical. If M is cofinitely* Rad*-supplemented such that* Rad*-supplements are supplements in M, then M is cofinitely supplemented.*

Proof. It can be seen from Lemma 2.7. □

Corollary 2.9. *Let R be any ring. If the R-module R is* Rad*-supplemented such that* Rad*-supplements are supplements in R, then R is semiperfect.*

Proposition 2.10. *Let M be an R-module. If every cofinite submodule U of M has a* Rad*-supplement V in M such that $U \cap V$ has a supplement in V, then M is cofinitely supplemented.*

Proof. Let U be any cofinite submodule of M. By assumption, there is a submodule V in M such that V is a Rad-supplement of U in M and $U \cap V$ has a supplement X in V. Then $U \cap V + X = V$ and $(U \cap V) \cap X = U \cap X \ll X$. Now

$$M = U + V = U + U \cap V + X = U + X.$$

Hence X is a supplement of U in M and it follows that M is cofinitely supplemented. □

3. Amply Cofinitely Rad-Supplemented Modules

In this section, we define the concept of amply cofinitely Rad-supplemented modules and give various the properties of them. We show that a module M is amply cofinitely Rad-supplemented if and only if every maximal submodule of

M has ample Rad-supplements in M and any ring R is left Rad-supplemented if and only if every left R-module is amply cofinitely Rad-supplemented. In addition, we give a characterization of generalized semiperfect modules.

Definition 3.1. A module M is *amply cofinitely* Rad-*supplemented* if every cofinite submodule of M has ample Rad-supplements in M.

It is clear that every amply Rad-supplemented module is amply cofinitely Rad-supplemented.

Proposition 3.2. *Let M be an R-module. If M is an amply cofinitely* Rad-*supplemented module, then every factor module of M is also an amply cofinitely* Rad-*supplemented module.*

Proof. Let U be any submodule of M and let X/U be a cofinite submodule of M/U. Suppose that $X/U + Y/U = M/U$ for some submodule Y/U of M/U. Then $X + Y = M$. Since X is cofinite and M is amply cofinitely Rad-supplemented, there is a submodule V of Y such that V is a Rad-supplement of X in M. By Lemma 2.3, $(V + U)/U$ is a Rad-supplement of X/U in M/U. Clearly Y/U contains $(V + U)/U$. Hence M/U is amply cofinitely Rad-supplemented. □

Corollary 3.3. *A homomorphic image of an amply cofinitely* Rad-*supplemented module is an amply cofinitely* Rad-*supplemented.*

Let M be an R-module and N be a submodule of M. $P(N)$ will indicate the collection of maximal submodules U of M with $N \subseteq U$.

Büyükaşık and Lomp [4] call a module is *w-local* if it has a unique maximal submodule, and proved that any Rad-supplement of a maximal submodule is w-local. Morever w-local modules are cofinitely Rad-supplemented. Now using this fact we can prove the following lemma.

Lemma 3.4. *Let M be an R-module. Then every maximal submodule of M has ample* Rad-*supplements in M if and only if $P(Rm) = P(cgs(Rm))$ for every element m in M-*Rad$M$*, where cgs(Rm) is the sum of all cofinitely* Rad-*supplemented submodules of Rm.*

Proof. Let m be any element in M-Rad$M$ and let U be a maximal submodule of M such that $Rm \nsubseteq U$. Then $U + Rm = M$. By assumption, there is a submodule K of M such that K is Rad-supplement of U in M and $K \subseteq$cgs(Rm). Hence $K \nsubseteq U$ and $P($cgs$(Rm))$=$P(Rm)$. Conversely, suppose that U is a maximal submodule of M and K is a submodule of M such that $U + K = M$. Then there is $m \in K$ such that $m \notin U$ hence $m \notin M$-Rad$M$. Note that cgs(Rm) is cofinitely Rad-supplemented. By assumption, U+cgs$(Rm) = M$ and it follows that U has a Rad-supplement V in M such that $V \subseteq$cgs$(Rm) \subseteq K$. Hence U

has ample Rad-supplements in M. □

Recall from [2] a submodule U of M has *ample supplements* in M if every submodule V of M such that $U + V = M$ contains a supplement of U in M. The module M is called *amply cofinitely supplemented* if every cofinite submodule has ample supplements in M. Following [2], a module M is amply cofinitely supplemented if and only if every maximal submodule of M has ample supplements in M (see Theorem 2.10). The following is an amply cofinitely Rad-supplemented module analogue of this fact.

Theorem 3.5. *Let M be an R-module. The following statements are equivalent.*

(a) M is amply cofinitely Rad-*supplemented.*

(b) Every maximal submodule of M has ample Rad-*supplements in M.*

*(c) $P(Rm) = P(\mathrm{cgs}(Rm))$ for every element m in M-*Rad$M$*, where cgs(Rm) is the sum of all cofinitely* Rad-*supplemented submodules of Rm.*

Proof. (a)$\Longrightarrow$(b) It is clear.

(b)$\Longrightarrow$(a) Let N be a cofinite submodule of M and L be a submodule of M such that $M = N + L$. Suppose that $M \neq N+\mathrm{cgs}(L)$. Then, since $N+\mathrm{cgs}(L)$ is a cofinite submodule of M, there is a maximal submodule U of M such that $N+\mathrm{cgs}(L)\subseteq U$. Since U is maximal, there is $m \in L$ such that $m \notin U$. Then $U + Rm = M$. By (b) there is a submodule K of M such that K is a Rad-supplement of U in M and it follows that K is a cofinitely Rad-supplemented module. Hence $K \subseteq\mathrm{cgs}(L)$. Then $U = M$. This is a contradiction because U is maximal. Hence $M = N+\mathrm{cgs}(L)$. Note that

$$cgs(L)/(N \cap cgs(L)) \cong M/N.$$

$N\cap\mathrm{cgs}(L)$ is a cofinite submodule of $\mathrm{cgs}(L)$. Let V be a Rad-supplement of $N\cap\mathrm{cgs}(L)$ in $\mathrm{cgs}(L)$. It follows that

$$M = N + cgs(L) = N + N \cap cgs(L) + V = N + V$$

and

$$N \cap V = (N \cap cgs(L)) \cap V \subseteq \mathrm{Rad}V.$$

Hence N has a Rad-supplement V in M such that $V \subseteq\mathrm{cgs}(L) \subseteq L$.

(b)$\Longleftrightarrow$(c) is proved Lemma 3.4. □

Corollary 3.6. *Let M be finitely generated an R-module. If every cyclic submodule of M is a cofinitely* Rad-*supplemented module, then M is amply cofinitely* Rad-*supplemented.*

Theorem 3.7. *The following satements are equivalent for a ring R:*

(a) R is left Rad-*supplemented.*

(b) Every left R-module is cofinitely Rad-*supplemented.*

(c) Every left R-module is amply cofinitely Rad-*supplemented.*

Proof. (a)$\Longleftrightarrow$(b) is proved in [4].

(a)$\Longrightarrow$(c) Let M be any left R-module. If R is a left Rad-supplemented ring, for every $m \in M$, Rm is a cofinitely Rad-supplemented and it follows that $P(Rm) = P(\text{cgs}(Rm))$. By Theorem 3.5, M is an amply cofinitely Rad-supplemented module.

(c)$\Longrightarrow$(b) Since an amply cofinitely Rad-supplemented module is cofinitely Rad-supplemented, the proof is clear. □

We give an example of module, which is cofinitely Rad-supplemented, but not amply cofinitely Rad-supplemented.

Example 3.8. The $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_p$, p any prime, is cofinitely Rad-supplemented, but it is not amply cofinitely Rad-supplemented. If the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_p$ is an amply cofinitely Rad-supplemented, the maximal submodule $\mathbb{Q}\oplus 0$ of $\mathbb{Q} \oplus \mathbb{Z}_p$ has ample Rad-supplements in $\mathbb{Q} \oplus \mathbb{Z}_p$ by Theorem 3.5. Since $\mathbb{Z}_p$ is simple, $\mathbb{Z}_p = \mathbb{Z}a$ for every nonzero element a in $\mathbb{Z}_p$. Let $0 \neq b \in \mathbb{Q}$. Then $\mathbb{Q} \oplus \mathbb{Z}_p = \mathbb{Z}(b+a) + \mathbb{Q}\oplus 0$. By assumption, there is a w-local submodule V of $\mathbb{Z}(b+a)$ with $\mathbb{Q} \oplus \mathbb{Z}_p = V + \mathbb{Q}\oplus 0$. Note that finitely generated w-local modules are local. Since $\mathbb{Z}$ is non-local noetherian ring, $\mathbb{Z}$ does not have a w-local ideal. Therefore $\mathbb{Z}(b+a)$ is not isomorphic to $\mathbb{Z}$ and it follows that there is a nonzero n in $\mathbb{Z}$ such that $n(b+a) = 0$, and so $nb = 0$. This is a contradiction since $\mathbb{Q}$ is not torsion.

The R-module M is called *π-projective* if, for any two submodules U, V of M with $U+V = M$, there is $f \in End(M)$ with $\text{Im}f \subseteq U$ and $\text{Im}(1-f) \subseteq V$ [5]. The proof of the following proposotion is the same as Theorem 2.15 in [6].

Proposition 3.9. *Let M be a π-projective cofinitely* Rad-*supplemented module. Then M is an amply cofinitely* Rad-*supplemented module.*

Proof. Let N be any cofinite submodule of M and let K be a submodule of M with $M = N + K$. Since M is π-projective, there is $f \in End(M)$ with $\text{Im}f \subseteq N$ and $\text{Im}(1-f) \subseteq K$. Suppose that V is a Rad-supplement of K in M. Now

$$M = f(M) + (1-f)(M) = f(M) + f(N+V) \subseteq N + (1-f)(V) \subseteq M\,,$$

so that $M = N + (1-f)(V)$. Note that $N + (1-f)(V)$ is a submodule of K. Let m be an element in $N \cap (1-f)(V)$. Then $m \in N$ and $m = (1-f)(n) = n - f(n)$ for some $n \in V$. Next $n = m + f(n) \in N$, so that $m \in (1-f)(N \cap V)$. But

$N \cap V \subseteq \mathrm{Rad}V$ gives that $N \cap (1-f)(V) = (1-f)(N \cap V) \subseteq \mathrm{Rad}((1-f)(V))$. Thus $(1 - f(V)$ is a Rad-supplement of N in M. It follows that M is an amply cofinitely Rad-supplemented module. □

The following statement gives an easy modification of Theorem 2.8.

Theorem 3.10. *Let M be an R-module with small radical. If M is amply cofinitely* Rad-*supplemented such that* Rad-*supplements are supplements in M, then M is amply cofinitely supplemented.*

Recall from [3] that an epimorphism $\alpha : P \to M$ is called a *generalized cover* if $\ker \alpha \subseteq \mathrm{Rad}P$, and a generalized cover $\alpha : P \to M$ is called *generalized projective cover* in case P is projective. Every small cover is a generalized cover, but a generalized cover need not to be a small cover. For example, $\pi : M \to M/\mathrm{Rad}M$ is a generalized cover but it is a not small cover, for a module M with $\mathrm{Rad}M$ non small. In [3], a module M is called *generalized semiperfect* if every factor module of M has a generalized projective cover. The concepts of generalized semiperfect modules were introduced in [3].

Azumaya proved (see [3], Theorem 4) that M generalized semiperfect if and only if each proper submodule of M is contained in a maximal submodule of M and each simple factor module of M has a generalized projective cover. We give the next characterization of generalized semiperfect modules thank to ample Rad-supplements of maximal submodules. Firstly we need the following lemma.

Lemma. 3.11. *Let M be an R-module. Then every maximal submodule of M has ample* Rad-*supplements (in M) which have a generalized projective cover if and only if every simple factor module of M has a generalized projective cover.*

Proof. Let M/N be any simple factor module of M. Then N is a maximal submodule of M. By assumption, N has a Rad-supplement K which has a generalized projective cover. Let $\alpha : P \to K$ be a generalized projective cover. Now

$$K \to K/K \cap N \cong (N + K)/N = M/N.$$

Then it is easy to see M/N has a generalized projective cover. Conversely, suppose that N is any maximal submodule of M and $M = N + K$, for some submodule K of M. If $\alpha : P \to M/N$ is a generalized projective cover,

$$f : K \to M/N\,, \qquad k \to k + N$$

is an epimorphism. Since P is projective, there is a homomorphism $\beta : P \to K$ such that $f\beta = \alpha$. Then $M = N + \beta(P)$ and $N \cap \beta(P) \subseteq \mathrm{Rad}\beta(P)$. Hence $\beta(P) \subseteq K$ is a Rad-supplement of N in M and $\beta(P)$ has a generalized projective cover. □

Theorem 3.12. *Let M be an R-module. M is generalized semiperfect if and only if every proper submodule of M is contained in a maximal submodule of M and every maximal submodule of M has ample* Rad*-supplements which have a generalized projective cover.*

Proof. It can be seen from Lemma 3.11 and Theorem 4 in [3]. □

Corollary 3.13. *Let M be a finitely generated module. Then, M is generalized semiperfect if and only if every maximal submodule of M has ample* Rad*-supplements which have a generalized projective cover.*

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