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MULTI-LAYER NATURE OF MATRIX SOLUTIONS OF NONLINEAR WAVE EQUATIONS

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Abstract: The whole spectrum of angular parameter of matrix solutions of nonlinear wave equations is defined and multi-layer nature of matrix solutions is considered. It is shown that reduction of matrix solutions leads to contracting the spaces and composing the dimensions. Matrix solutions are used as operators of rotations to show that any vector in the matrix space can be turned to the given basis vector. A spiral of evolution is easily modelled using multi-layer nature of matrix solutions. These matrices are useful also to model a collision of helical strings, for description of vortex rings and their collision, for definition of energetic levels.

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1. Introduction

More perfect description of matrix solutions of nonlinear wave equations is given in this paper. First of all we consider the whole spectrum of values of angular parameter. In the previous papers [3] and [5] we have used the principal value of angular parameter ϕ, $0 \leq \phi \leq \pi$. Now the following form of angular parameter is considered $\Phi = \phi + m\pi$, where m is an arbitrary integer. This form allows us to model multi-layer structures as ladder evolution in conformity to particles.

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Next, we give more perfect definition of the reduction of matrix solutions, which leads to contracting the spaces and to composing the dimensions. Each step of the reduction is connected with passage from one frequency level of rotations to another level with doubling the angular velocity of rotation.

Further, we prove that any vector in many dimensional matrix space can be turned in such a way that its expansion on basis matrices includes only diagonal matrices, in particular, any vector can be turned to the given basis vector.

We also mention a usefulness of matrix solutions for modelling the collision of particles, for description of ring vortices and their collision, and for definition of energetic levels in the form of concentric circles.

2. Basic Constructions

Consider the Klein-Gordon nonlinear wave equation, written in the natural system of units $\hbar = c = 1$

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{dQ}{du} = 0, \qquad Q(u) = \frac{\mu^2}{4}(u^2 - 1)^2. \tag{1}$$

The corresponding sequence of matrix solutions u_n $(n = 2^k, k = 1, 2, \ldots)$ for this equation is constructed as follows

$$u_n(\Phi, \mathbf{a}) = \cos(\Phi)E_n + \mathbf{a}\sin(\Phi) = \exp(\Phi\mathbf{a}), \; \mathbf{a} = \sum_{j=1}^{2k+1} a_j M_j, \; |\mathbf{a}| = 1. \tag{2}$$

The whole spectrum of values of angular parameter is given by the following form:

$$\Phi = \phi + m\pi, \quad \phi = \operatorname{arccot}(-\sinh(\alpha z)), \quad \alpha = \mu\sqrt{2/(1 - v^2)}, \tag{3}$$

where m is arbitrary integer, $z = x - vt$ is the moving frame of reference, x is an arbitrary point along the chosen radial direction in 3D-space of coordinates x_1, x_2, x_3.

Other notation: E_n is the unit $n \times n$-matrix, matrices M_j are unitary, $M_j M_j^* = E_n$ ($*$ means transposition and conjugation), anti-Hermitian, $M_j^* = -M_j$, anti-commuting and linearly independent $n \times n$-matrices.

Note that angular parameter ϕ for the nonlinear wave equations behaves as a kink-function of variable z in the region $-\infty \le z \le \infty$, so that ϕ defines the initial (first) layer $0 \le \phi \le \pi$. Angular parameter $\Phi = \phi + m\pi$ defines m-layer and maps 3D+1-space into this layer. Due to radial direction in 3D-space of coordinates x_1, x_2, x_3 can be defined by direction cosines, we put $x =$

$\sum x_j \cos(\theta_j)$ and find that function $\phi(z) \equiv \phi(x_1, x_2, x_3, t)$ maps each point of 3D+1-space into the initial layer. For fixed z there exists only one point $x = z/(1+v^2), t = -zv/(1+v^2)$ on the line $x + t/v = 0$, hence, this line (perpendicular to line $x - vt = 0$) can be taken as z-axis.

For a linear wave equation, for example, for equation (1) with potential $Q_0 = -\mu^2 u^2 + Const$, the angular parameter of matrix solution $u_n(\phi, \mathbf{a})$ is a linear function $\phi = \alpha z$. Therefore, only one layer $-\infty \le \phi \le \infty$ exists in this case.

Now we propose the following construction of $2k+1$ anti-commuting matrices $M_j(n) \equiv M_j$ of the order $n = 2^k$ $(k = 1, 2, \ldots)$.

Theorem 1. *For $k = 1$ let be $M_j(2) = H_j$ $(j = 1, 2, 3)$, where H_j are unit quaternions*

$$H_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and for $k = 2, 3, \ldots$ let there be

$$M_j(n) = \begin{pmatrix} M_j(n/2) & 0 \\ 0 & M_j^*(n/2) \end{pmatrix}, \qquad j = 1, 2, \ldots, 2k-1, \tag{4}$$

$$M_{2k}(n) = \begin{pmatrix} 0 & -E_{n/2} \\ E_{n/2} & 0 \end{pmatrix}, \quad M_{2k+1}(n) = \begin{pmatrix} 0 & iE_{n/2} \\ iE_{n/2} & 0 \end{pmatrix}. \tag{5}$$

Then these matrices are unitary, anti-Hermitian, anti-commuting, and linearly independent.

The proof is omitted. It can be easily derived by mathematical induction. Matrices M_j of arbitrary even and odd order are constructed in [5] as the diagonal block matrices.

Anti-commuting, anti-Hermitian, unitary, and linearly independent matrices $M_1, M_2, \ldots, M_{2k}$ can be taken as the generators of associative 2^{2k}-dimensional Clifford algebra $Cl_{0,2k}$ with the basis

$$E_n, M_1, \ldots, M_{2k}, M_1 M_2, \ldots, M_1 M_2 \ldots M_{2k}, \quad i < j, \ i, j = 1, 2, \ldots, 2k. \tag{6}$$

In this paper, Clifford algebra $Cl_{0,2k}$ over reals R is generated by matrices $M_1, \ldots, M_{2k}$ and is considered as 2^{2k}-dimensional vector space and denoted for convenience by $V(n)$. Elements of space $V(n)$ are $n \times n$-matrices, in particular, matrix solution u_n.

Note, that matrix M_{2k+1} is equal, up to i^r, to the last element of the basis (6), namely $M_{2k+1} = i^r \prod_{j=1}^{2k} M_j$, where r is a remainder of $k/4$. There is no other product $M_{i_1} \ldots M_{i_m}$ $(1 < m < 2k, i_1 < \ldots < i_m)$ that anti-commutes with each of M_j for all $j = 1, 2, \ldots, 2k$. Therefore, the construction of solutions u_n of nonlinear wave equations is based precisely on the complete system of $2k+1$ anti-commuting matrices M_j of the order $n = 2^k$.

Matrix solution $u_n(\Phi, \mathbf{a})$, being considered as an operator of rotation, rotates unit vector $\mathbf{b} = \sum b_j M_j$ (perpendicular to $\mathbf{a}$) about $\mathbf{a}$ according to the formula

$$u_n(\Phi, \mathbf{a})\mathbf{b} = \mathbf{b}\cos\Phi + \mathbf{ab}\sin\Phi, \qquad \sum a_j b_j = 0.$$

It is evident that vector $\mathbf{b}$ accomplishes a turn about vector $\mathbf{a}$ by angle Φ in the plane spanned over vectors $\mathbf{b}$ and $\mathbf{ab}$. It follows from the anti-commuting property of matrices M_j that $\mathbf{ab} = -\mathbf{ba}$ and, hence, it is easy to show that three vectors $\mathbf{a}, \mathbf{b}, \mathbf{ab}$ form an orthogonal frame. Therefore, we often call matrix u_n itself a rotation, without indicating a vector on which u_n acts. We also say that rotation u_n forms a rotation field.

3. Reduction

In this section we describe a procedure of reduction of rotation u_n to rotation $u_{n/2}$ and sequentially till u_1. The reduction of rotations closely accompanied by contracting the space $V(n)$ to space $V(n/2)$ and sequentially till the complex plane $V(1)$. This process also can be treated as a composing the dimensions due to the number of dimensions is decreased in 4 times on each step of the reduction.

Let us note some properties of matrix solution u_n

$$u_n^*(\Phi, \mathbf{a}) = u_n(-\Phi, \mathbf{a}) = u_n(\Phi, -\mathbf{a}) = u_n^{-1}(\Phi, \mathbf{a}) \tag{7}$$

as well as

$$u_n^2(\Phi, \mathbf{a}) = u_n(2\Phi, \mathbf{a}), \qquad \mathbf{a}^2 = -E_n. \tag{8}$$

Matrices $M_j(n)$ for $j = 1, 2, \ldots, 2k-1$ have block diagonal form and, therefore, matrix u_n also has block diagonal form

$$u_n(\Phi, \mathbf{a}) = \mathrm{diag}(u_{n/2}(\Phi, \mathbf{a}), u_{n/2}^*(\Phi, \mathbf{a})), \qquad \mathbf{a} = \sum_{j=1}^{2k-1} a_j M_j. \tag{9}$$

Further, as follows from the theorem about composition and decomposition of simultaneous rotations [5], the composition of two simultaneous rotations gives one whole rotation as follows

$$u_n(p\Phi, \mathbf{a}) \otimes u_n(q\Phi, \mathbf{b}) = u_n(l\Phi, \mathbf{c}), \qquad l\mathbf{c} = p\mathbf{a} + q\mathbf{b}, \tag{10}$$

where $\mathbf{c}, \mathbf{a}, \mathbf{b}$ are unit vectors and l, p, q are arbitrary integers.

Using (10) we find

$$u_{n/2}(2\Phi, \mathbf{a}) \otimes u_{n/2}(-\Phi, \mathbf{a}) = u_{n/2}(\Phi, \mathbf{a}).$$

Let us rewrite this equality, taking into account (7), in the form

$$u_{n/2}(2\Phi, \mathbf{a}) = u_{n/2}(\Phi, \mathbf{a})/u^*_{n/2}(\Phi, \mathbf{a}).$$

Namely this form we use for the definition a transformation of the reduction

$$u_n(\Phi, \mathbf{a}) \to u_{n/2}(\Phi, \mathbf{a})/u^*_{n/2}(\Phi, \mathbf{a}) = u_{n/2}(2\Phi, \mathbf{a}), \tag{11}$$

which we call the linear fractional transformation. This transformation of diagonal matrix u_n looks as overturn of lower block $u^*_{n/2}$ and composing it with upper block $u_{n/2}$. The process of reduction can be continued as follows

$$u_n(\Phi, \mathbf{a}) \to u_{n/2}(2\Phi, \mathbf{a}) \to \dots u_2(\Phi n/2, \mathbf{a}) \to u_1(n\Phi), \tag{12}$$

However, vector $\mathbf{a}$ should be represented as an expansion along diagonal basis matrices on each step of the reduction. This representation can be achieved by corresponding turn of vector $\mathbf{a}$ to given vector-matrix M_0 of the block diagonal form. It is appropriately to construct matrix M_0 on the base of the first basis matrices M_j, assuming that vector $\mathbf{a}$ has nontrivial components along the others M_j for to have nonzero angle of rotation. Matrix u_n is used as an operator of rotation to accomplish this turn.

Theorem 2. *For given unit vector* $\mathbf{a}$ *let be*

$$M_0 = \sum_{j=1}^{l} \lambda_j M_j, \quad \sum_{j=1}^{l} \lambda_j^2 = 1, \quad 1 \le l < 2k,$$

assuming that $0 < \sum_{j=l+1}^{2k+1} a_j^2 \le 1$. *Further, let angle of rotation be defined as follows:* $\theta = \pi/2$ *in the case* $a_j = 0$ *for* $j = 1, \dots, l$, *otherwise* θ *is found from the relation* $0 < \cos^2\theta = \sum_{j=1}^{l} a_j^2 < 1$ *and in this case we put* $\lambda_j = a_j/\cos\theta$. *Moreover, let be* $\mathbf{b} = \sum_{j=l+1}^{2k+1} a_j M_j M_0/\sin\theta$. *Then rotation* $u_n(\theta, \mathbf{b})$ *turns vector* $\mathbf{a}$ *to the vector* M_0.

Proof. Consider the rotation $u_n(\theta, \mathbf{b})$, taking into account the anti-commuting property of matrices M_j and equality $M_j^2 = -E_n$. We have

$$\mathbf{a} \to u_n(\theta, \mathbf{b})\mathbf{a} = \mathbf{a}\cos\theta + \mathbf{b}\mathbf{a}\sin\theta = M_0\cos^2\theta + \sum_{j=l+1}^{2k+1} a_j M_j \cos\theta$$

$$+ \sum_{j=l+1}^{2k+1} a_j M_j M_0 (M_0\cos\theta + \sum_{j=l+1}^{2k+1} a_j M_j) = M_0\cos^2\theta + M_0 \sum_{j=l+1}^{2k+1} a_j^2 = M_0.$$

Theorem 2 is proved. □

Corollary 1. *Let be $M_0 = M_l$ and $a_l = 0$ for given $l \in \{1, \ldots, 2k-1\}$ then $\theta = \pi/2$. Let be also $\mathbf{b} = \sum_{j\neq l}^{2k+1} a_j M_j M_0$. Then $\mathbf{a} \to u_n(\theta, \mathbf{b})\mathbf{a} = M_l$.*

The last step in the chain of reduction (12) can be fulfilled if $\mathbf{a}$ is turned to M_3 so that M_3 would be converted to H_3 in the space $V(2)$. Then we obtain matrix $u_2(\Phi n/2, H_3) = \mathrm{diag}(\exp(i\Phi n/2), \exp(-i\Phi n/2))$, which is converted to $u_1 = \exp(in\Phi)$ by use the linear fractional transformation. Thus the last step of reduction leads to circular motion with maximal angular velocity.

4. Layers and Levels

Multi-layer nature of matrix solutions can be used for description a spiral of evolution. Let us consider the simplest solution u_1 of the nonlinear wave equation in conformity to particles. On the initial layer, $0 \leq \Phi \leq \pi$, this solution describes a half-turn from state $+1$ to state -1. On the next layer, $\pi \leq \Phi \leq 2\pi$, it describes a half-turn from state -1 to state $+1$ etc. Each half-turn is accomplished when z varies from $-\infty$ to ∞, i.e. during the lifetime of the particle. On the next layer the particle arises as a new-born particle and etc.

Note that kink-function ϕ has a derivative in the form of hump in the neighborhood of $\phi = \pi/2$ and asymptotically diminishes to zero for $z \to \pm\infty$. It means that the process of transferring from one state to another goes non-uniformly. The particle infinitely long time is situated in the neighborhood of layer boundary and very quickly goes through middle zone of the layer.

It should be noted that for linear wave equation solution $u_1 = \exp(i\phi)$ with $\phi = \alpha z$ is represented as a helix with infinitely many coils along z-axis. Thus in the linear case solution u_1 describes infinitely many turns with constant angular velocity in the time when z varies from $-\infty$ to ∞. It is clear that this solution is fit for description uniform rotations, wave processes, helical strings.

When we use matrix solution $u_n(\Phi, \mathbf{a})$ for description of rotations we assume that u_n rotates the certain, perpendicular to vector $\mathbf{a}$, unit vector $\mathbf{b}$ about $\mathbf{a}$ by angle Φ. For a set of unit vectors $\mathbf{a}$ this rotation forms a sphere, say S_n, spanned over the frame $M_1, \ldots, M_{2k+1}$. Therefore, we can say that the chain (12) is accompanied by sequence of spheres $S_n, S_{n/2}, \ldots, S_1$. We assume that all these spheres have one and the same origin.

Let us estimate the radius of each sphere. According to Corollary 1 each unit vector $\mathbf{a}$ (unit vector in the sense of $\sum a_j^2 = 1$) can be turned to one of the basis vectors. Therefore, it is sufficient to estimate the norms of basis matrices M_j.

Theorem 3. *The quadratic norm of matrix $M_j(n)$ $(j = 1, \dots, 2k+1)$ is equal to $\sqrt{n}$ and the radius of sphere S_n is also equal to $\sqrt{n}$; each consequent sphere has the radius $\sqrt{2}$ times smaller than the radius of the previous sphere; spheres are embedded each to another and the smallest S_1 is a unit circle.*

Proof. It is evident that quadratic norm of basis elements, E_2, H_1, H_2, H_3, of space $V(2)$ is equal to $\sqrt{2}$. Assume that quadratic norm of any $M_j(n/2)$ for $j = 1, \dots, 2k-1$ is equal to $\sqrt{n/2}$. Then the quadratic norm of $M_j(n)$ for the same j is equal to

$$||M_j(n)|| = ||\operatorname{diag}(M_j(n/2), M_j^*(n/2))|| \\ = \sqrt{||M_j(n/2)||^2 + ||M_j^*(n/2)||^2} = \sqrt{n}\,.$$

It is easy to see that matrices $M_{2k}(n)$ and $M_{2k+1}(n)$ also have norm $\sqrt{n}$. Thus the radius of sphere S_n is equal to $\sqrt{n}$ and the radius of sphere $S_{n/2}$ is equal to $\sqrt{n/2}$ and so on till sphere S_2 with radius $\sqrt{2}$ and circle S_1 of unit radius. The theorem is proved. □

Each step of the reduction corresponds to rotation with definite angular velocity. For example, on p-step, where $p = 2^l (l = 1, 2, \dots, k)$, solution $u_{n/p}(p\Phi, \mathbf{a})$ acts in pm-layer and describes the rotation with angular velocity $\omega = pd\Phi/dt$. We can put into correspondence the sequence of spheres to sequence of levels of rotation frequency. It is naturally to represent these levels by circles in the equatorial plane $V(1)$.

Circle of radius $\sqrt{n/p}$ corresponds to sphere $S_{n/p}$, where rotations are accomplished with angular velocity $\omega = pd\Phi/dt$. The maximal frequency of rotations is reached on the circle S_1, where $\omega = nd\Phi/dt$. Thus, each step of the reduction describes the transferring from one level of frequency to another with simultaneous doubling of rotation angle. This levels of rotation frequency we associate with energy levels.

An energy is proportional to square of linear velocity, which is defined as $v = \omega 2\pi R$, where R is the radius of corresponding circle. It follows both from the doubling of the angular velocity and from the decreasing of the radius of sphere in $\sqrt{2}$ times that linear velocity increases in $\sqrt{2}$ times on each step of the reduction. Thus, energy of rotation on each step increases in 2 times and reaches the maximal value, proportional to $n^2(d\phi/dt)^2$, on the smallest circle S_1. This process can be represented by ascending contracting spiral along the vertical energy-axis since energy increases in 2 times but radii of coils decrease in $\sqrt{2}$ times on each step.

5. Particles Collision and Vortices

Green, Schwarz and Witten [2] have proposed a fine model of superstrings collision which imitates the particles collision. It looks that this model is precisely created to use matrix solutions u_n in the form of helical strings. Indeed, in the linear case matrix solution $u_n(\phi, \mathbf{a}) = \exp(\phi\mathbf{a})$ for $\phi = \alpha z$ accomplishes rotation with infinitely many turns about vector $\mathbf{a}$. In this case solution u_n would be better to represent geometrically by helical string with infinitely many coils along $\mathbf{a}$-direction. It is clear that the helical string can be compressed to a thin ring or can be expanded to infinitely long helix. Assuming that helical strings in the form of thin rings imitate the particles we propose the following scheme of their collision.

Let two or more helical strings imitate two or more particles and let point O be a point of their intersection. Since each helical string is characterized by its own vector $\mathbf{a}$ – rotation axis and direction of motion, we conclude that point O is a point of intersection of corresponding $\mathbf{a}$-directions. Considering a neighborhood of point O, i.e. sphere S_0 spanned over helical rings with origin O, we use the theorem of composition and decomposition of simultaneous rotations [5]. Each helical string is a rotation about its own rotation vector $\mathbf{a_j}$. Composition of simultaneous rotations gives the whole rotation about certain vector $\mathbf{a_0} = \sum \mathbf{a_j}$. Thus there exists a single whole rotation in the sphere S_0, which can be decomposed onto several separate rotations, corresponded to outgoing particles. The latter are imitated by helical strings with another vectors $\mathbf{a_i}$ such that $\sum \mathbf{a_i} = \mathbf{a_0}$.

Now, consider the models of toroidal vortex rings arising in fluid and other flows. Atsukovsky [1] gives detailed and exhaustive description of stable toroidal vortices in gas-like ether by use of gas and fluid dynamics. Moreover, he considers vortical models of basic elementary particles such as proton, neutron, electron. In [4] we have made an attempt to use matrix solutions for describing the models of toroidal vortices arising both in the fluid flow behind the acute edge and in the electron flow behind the acute cathode. The essence of our construction of the vortical model is consisted from a possibility to describe one helical motion around another using a pair of matrix solutions.

Let us take a helical string compressed into a thin ring and consider the second helical string winded around the ring. As the result we obtain a toroidal form. This form is more stable than the ring, considered above for an imitation a particle, and namely the toroidal form would be better to use for an imitation the particles. Note that toroidal form provides two different motions: circular motion along the ring and toroidal motion along the helical winding around this ring. These motions, being simultaneous, lead to a situation when each point on this toroidal object participate in two mentioned motions.

Each matrix solution $u_n(\phi, \mathbf{a}) = \exp(\phi\mathbf{a})$ of linear wave equation can be reduced by procedure (12) to simplest solution $\exp(n\phi)$, which can be represented as a ring or as a helical string. The arbitrary large number n shows how large is the angular velocity of rotations along the ring or along the coils of the helical string. Now we deal with two reduced solutions in the following way: one solution we represent by a ring but the second one – by helical winding around the first ring. The latter is possible, provided that rotation vector of the second solution (vector of angular velocity) is tangent to the ring and moves along the ring.

Since all points of the toroidal object participate in the circular motion, we can consider a collision of two such objects (or particles) in the same way as above and can obtain a composed toroidal object. However, if these two objects have opposite rotation vectors, $\mathbf{a}$ and $-\mathbf{a}$, then the composition of rotations leads to zero rotation vector, i.e. leads to annihilation. In the reality it can be represented as a rupture of two rings and formation two helical strings in consequence of their straightening after the rupture. These two helical strings with huge angular velocity along their coils looks like mini tornado. They can flight out in opposite directions or they remain in the sphere of their collision.

The latter can be explained as follows: if two helical strings are in the contact with each other as two rotating tubes with parallel axes, provided opposite rotations along their coils, then linear velocities at the points of contact have the same direction. Hence, the pressure between tubes is at least two times smaller than the pressure on the external sides of these tubes. Thus, tubes (helical strings) attract each other and form a stable pair, which does not have a single whole rotation but each element of the pair has its own helical motion.

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