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A HILBERT-TYPE INEQUALITY WITH PARAMETER

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Abstract: In this paper it is shown that a Hilbert-type inequality with parameter λ can be established by introducing a real functions $u(x)$. In particular, two forms of the Hilbert-type inequality are combined into one similar form, when $u(n) = n+\frac{\lambda}{2} \neq 0$, where $n \in N_0$ and λ is an integer. As applications, some refinements on the Fejer-Riesz type inequality and the Hardy type inequality are given.

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1. Introduction

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. If $\lambda = 0, 1$, then

$$\left|\sum_{m=1-\lambda}^{\infty}\sum_{n=1-\lambda}^{\infty}\frac{a_m\bar{b}_n}{m+n+\lambda}\right|^2 \leq \pi^2 \sum_{n=1-\lambda}^{\infty}|a_n|^2 \sum_{n=1-\lambda}^{\infty}|b_n|^2, \tag{1.1}$$

and

$$\left|\sum_{m=1-\lambda}^{\infty}\sum_{\substack{n=1-\lambda,\\ m\neq n}}^{\infty}\frac{a_m\bar{b}_n}{m-n}\right|^2 \leq \pi^2 \sum_{m=1-\lambda}^{\infty}|a_n|^2 \sum_{n=1-\lambda}^{\infty}|b_n|^2, \tag{1.2}$$

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where the constant factor π^2 is the best possible. It is well-known that the inequalities (1.1) and (1.2) are called Hilbert Theorem for double series. The two forms (1.1) and (1.2) on the Hilbert inequality were combined into one similar form in some papers (such as [6]-[1] etc.), i.e.:

$$\left|\sum_{m=1-\lambda}^{\infty}\sum_{n=1-\lambda}^{\infty}\frac{a_m\bar{b}_n}{m+n+\lambda}\right|^2+\left|\sum_{m=1-\lambda}^{\infty}\sum_{\substack{n=1-\lambda,\\ m\neq n}}^{\infty}\frac{a_m\bar{b}_n}{m-n}\right|^2 \leq \pi^2\sum_{n=1-\lambda}^{\infty}|a_n|^2\sum_{n=1-\lambda}^{\infty}|b_n|^2. \quad (1.3)$$

Recently, the various extensions of (1.1) appeared in some papers (such as [2], [5], [7] etc.). They focalize on changing the denominator of the function of the left-hand side of (1.1). Such as the denominator $(m+n+\lambda)$ is replaced by $(\alpha_m+\beta_n)^\mu$ in the paper [2]; and the denominator $(m+n+\lambda)$ is replaced by $(mu\,(m)+nv\,(n))^\mu$ in the paper [5], etc. Some new results in these papers were yielded. The inequality (1.3) is obviously a significant refinement of (1.1) and (1.2). However it is few that the both extensions and refinements on (1.2) and (1.3) appear in previous papers. The main purpose of the present paper is to establish both an extension and a significant refinement on (1.3). Explicitly, let $u(x)>0$, $x\in[0,+\infty)$ be a real functions, $\lim_{x\to\infty}u(x)=+\infty$. $u(n)=Z_n+\frac{\lambda}{2}$, where Z_n indicates an integer ($n\in N_0,\lambda$ is a real number). If the denominator $(m+n+\lambda)$ of the first term of the left-hand side of (1.3) is replaced by $u(m)+u(n)$, and the denominator $(m-n)$ of the second term of the left-hand side of (1.3) is replaced by $u(m)-u(n)$, then a new inequality established is significant in theory and applications. In particular, as applications, we will give both extensions and refinements on the Fejer-Riesz inequality and the Hardy inequality. For convenience, we introduce some notations and functions:

$$U\,(a,b)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m\overline{b_n}}{u(m)+u(n)},$$

$$V\,(a,b)=\sum_{m=0}^{\infty}\sum_{n=0\,,\;u(m)\neq v(n)}^{\infty}\frac{a_m\overline{b_n}}{u(m)-u(n)}\,,$$

$$U_2\,(a,b)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m\overline{b_n}}{(u(m)+u(n))^2}\,,$$

$$\|x\|^2 = \sum_{n=0}^{\infty} |x_n|^2, \quad (f,g) = \int_0^{2\pi} f(t)\,\overline{g(t)}dt, \quad \|\alpha\|^2 = \int_0^{2\pi} |\alpha|^2\,dt\,,$$

where $\alpha = f, g$.

In particular, when $b = \bar{a}$, the notations $U(a,a), U_2(a,a)$ and $V(a,a)$ are denoted respectively by $U(a), U_2(a)$ and $V(a)$. And we stipulate that $u(n) = Z_n + \frac{\lambda}{2} \neq 0$, where Z_n indicates an integer ($n \in N_0, \lambda$ is an integer or $0 < \lambda < 1$). Throughout this paper we will frequently use these notations.

2. Lemmas

In order to prove our assertions we need the following lemmas.

Lemma 2.1. *If both* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are absolute convergent, then:*

(i) $\sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty} a_m \bar{b}_n$ *is absolute convergent.*

(ii) $\|a\|^2$ *and* $\|b\|^2$ *are convergent.*

The proof of it has been given in the paper [1]. Hence it is omitted here.

Lemma 2.2. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of complex numbers. Define a function by*

$$r(x) = \pi U(x) \sin 2\lambda\pi - U_2(x) \sin^2 \lambda\pi\,. \tag{2.1}$$

If λ *is an integer or* $\frac{1}{2} \leq \lambda < 1$, *then*

$$r(a)\, r(b) \geq 0 \tag{2.2}$$

Proof. When λ is an integer, it is clear that $r(a)\,r(b) = 0$. So we consider only the case for $\frac{1}{2} \leq \lambda < 1$. It is easy to deduce that

$$U(x) = \int_0^1 \left|\sum\nolimits_{m=0}^{\infty} x_m t^{u(m)-\frac{1}{2}}\right|^2 dt > 0,$$

$$U_2(x) = \int_0^1 \frac{ds}{s} \int_0^s \left|\sum\nolimits_{n=0}^{\infty} x_n t^{u(n)-\frac{1}{2}}\right|^2 dt.$$

When $\frac{1}{2} \leq \lambda < 1$, it is obvious that $r(x) < 0$. Hence we have $r(a)\,r(b) > 0$. □

We will frequently use the function $r(x)$, below.

Lemma 2.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^{u(n)-\frac{1}{2}}$. If $f(z)$ is analytic in the unit disc $|z| < 1$, then*

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} t \left| f\left(-e^{it}\right) \right|^2 dt \right| = |V(a)| . \tag{2.3}$$

Proof.
$$\left| \int_{-\pi}^{\pi} t \left| f(-e^{it}) \right|^2 dt \right| = \left| \int_{-\pi}^{\pi} t f(-e^{it}) \overline{f(-e^{it})} dt \right|$$

$$= \left| \int_{-\pi}^{\pi} t \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m \overline{a_n} \left(\cos(\pi + t) + i \sin(\pi + t)\right)^{u(m)-\frac{1}{2}} \right.$$

$$\left. \left(\cos(\pi - t) + i \sin(\pi - t)\right)^{u(n)-\frac{1}{2}} dt \right|$$

$$= 2\pi \left| \left(\sum_{m=0}^{\infty} \sum_{n=0,\ u(m)\neq u(n)}^{\infty} \frac{a_m \overline{a_n}}{u(m) - u(n)} \right) \sin \lambda\pi \right.$$

$$\left. + i \left(\sum_{m=0}^{\infty} \sum_{n=0,\ u(m)\neq u(n)}^{\infty} \frac{a_m \overline{a_n}}{u(m) - u(n)} \right) \cos \lambda\pi \right| = 2\pi |V(a)| .$$

Thereby the relation (2.3) holds. □

3. Main Results

We can apply the above lemmas to prove our assertions.

Theorem 3.1. *Let $r(x)$ be a function defined by (2.1), $\{a_n\}$ and $\{b_n\}$ be two nonzero sequences of complex numbers, and both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolute convergent. Then:*

(i) If λ is an integer, then

$$|U(a,b)|^2 + |V(a,b)|^2 \le \pi^2 \|a\|^2 \|b\|^2 . \tag{3.1}$$

(ii) If $0 < \lambda < 1$, then

$$\left| U(a,b) \cos 2\lambda\pi - \frac{1}{2\pi} U_2(a,b) \sin 2\lambda\pi \right|^2 + |V(a,b)|^2$$
$$\le \pi^2 \|a\|^2 \|b\|^2 - \frac{1}{\pi^2} r(a) r(b) . \tag{3.2}$$

In particular, when $\frac{1}{2} \leq \lambda < 1$*, we have* $r(a)\, r(b) > 0$*.*

Proof. Define two functions by

$$f(t) = \sum\nolimits_{m=0}^{\infty} a_m \sqrt{t} \sin(u(m))\, t\,, \quad g(t) = \sum\nolimits_{n=0}^{\infty} b_n \sqrt{t} \cos(u(n))\, t\,,\ \ t \in [0, 2\pi]\ .$$

Since the both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent, by Lemma 2.1, the double series $\sum\limits_{m=0}^{\infty} \sum\limits_{n=0}^{\infty} a_m \bar{b}_n$ is absolute convergent. Accordingly, $f(a,t)$ $g(b,t)$ is uniformly convergent in the interval $[0, 2\pi]$. Thereby the interchange in order of summation and integration can be made. Below we stipulate that the interchanges in order of summation and integration are justified. It is easy to deduce that

$$\|f\|^2 = \pi^2 \|a\|^2 - r(a)\,, \quad \|g\|^2 = \pi^2 \|b\|^2 + r(b)\ ,$$

$$\begin{aligned} |(f,g)| &= \left| \int_0^{2\pi} f(t)\, \overline{g(t)} dt \right| \\ &= \pi \left| V(a,b) + \left(U(a,b) \cos 2\lambda\pi - \frac{1}{2\pi} U_2(a,b) \sin 2\lambda\pi \right) \right| . \end{aligned}$$

We apply Cauchy-Schwarz's inequality to obtain

$$\begin{aligned} &\left| V(a,b) + \left(U(a,b) \cos 2\lambda\pi - \frac{1}{2\pi} U_2(a,b) \sin 2\lambda\pi \right) \right|^2 \\ &= \frac{1}{\pi^2} |(f,g)|^2 \leq \frac{1}{\pi^2} \|f\|^2 \|g\|^2 = \frac{1}{\pi^2} \left\{ \pi^2 \|a\|^2 - r(a) \right\} \left\{ \pi^2 \|b\|^2 + r(b) \right\} . \end{aligned} \tag{3.3}$$

(i) When λ is an integer, it is known from (2.1) that $r(x) = 0$. Hence the inequality (3.3) can be reduced to

$$|V(a,b) + U(a,b)|^2 \leq \pi^2 \|a\|^2 \|b\|^2 . \tag{3.4}$$

Notice that $U(b,a) = \overline{U(a,b)}$ and $V(b,a) = -\overline{V(a,b)}$. Then interchanging a and b in (3.4) we have

$$\left| -\overline{V(a,b)} + \overline{U(a,b)} \right|^2 \leq \pi^2 \|a\|^2 \|b\|^2 . \tag{3.5}$$

Adding (3.4) and (3.5), then the inequality (3.1) follows after simplifications

(ii) When $0 < \lambda < 1$, we obtain from (3.3)

$$\begin{aligned} &\left| V(a,b) + \left(U(a,b) \cos 2\lambda\pi - \frac{1}{2\pi} U_2(a,b) \sin 2\lambda\pi \right) \right|^2 \\ &\qquad \leq \pi^2 \|a\|^2 \|b\|^2 - \left(\|b\|^2 r(a) - \|a\|^2 r(b) \right) - \frac{1}{\pi^2} r(a)\, r(b)\ . \end{aligned} \tag{3.6}$$

Notice that $U(b,a) = \overline{U(a,b)}$, $U_2(b,a) = \overline{U_2(a,b)}$ and $V(b,a) = -\overline{V(a,b)}$. Then interchanging a and b in (3.6) we have

$$\left| -\overline{V(a,b)} + \left(\overline{U(a,b)} \cos 2\lambda\pi - \frac{1}{2\pi} \overline{U_2(a,b)} \sin 2\lambda\pi \right) \right|^2 \leq \pi^2 \|b\|^2 \|a\|^2 - \left(\|a\|^2 r(b) - \|b\|^2 r(a) \right) - \frac{1}{\pi^2} r(b) r(a) . \quad (3.7)$$

Adding (3.6) and (3.7), the inequality (3.2) can be gotten after simplifications. In particular, when $\frac{1}{2} \leq \lambda < 1$, by Lemma 2.2, we have $r(a) r(b) \geq 0$. The proof of the theorem is completed.

Corollary 3.1. *Let $\sum_{n=0}^{\infty} a_n$ be absolute convergent. Then:*

(i) If λ is an integer, then

$$|U(a)|^2 + |V(a)|^2 \leq \pi^2 \|a\|^4 . \quad (3.8)$$

(ii) If $0 < \lambda < 1$, then

$$\left| U(a) \cos 2\lambda\pi - \frac{1}{2\pi} U_2(a) \sin 2\lambda\pi \right|^2 + |V(a)|^2 \leq \pi^2 \|a\|^4 - \frac{1}{\pi^2} r^2(a) . \quad (3.9)$$

In particular, we suppose that $u(n) = n + \frac{\lambda}{2}$, when $\lambda = 0, 1$, we get (1.3) from (3.1) immediately. It follows that the inequality (3.1) is an extension of (1.3).

Corollary 3.2. *If $u(n) = n + \frac{1}{4}$, then*

$$\left| \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m \bar{b}_n}{m+n+\frac{1}{2}} \right) \right|^2 + \left| \sum_{m=0}^{\infty} \sum_{n=0, m\neq n}^{\infty} \frac{a_m \bar{b}_n}{m-n} \right|^2 \leq \pi^2 \|a\|^2 \|b\|^2 - \frac{1}{\pi^2} r(a) r(b) , \quad (3.10)$$

where $r(a) r(b) > 0$.

Clearly, it is known from Lemma 2.2 that $r(a) r(b) > 0$.

If $r(a) r(b)$ in (3.10) is replaced by zero, then the inequality (3.10) can be reduced to

$$\left| \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m \bar{b}_n}{m+n+\frac{1}{2}} \right) \right|^2 + \left| \sum_{m=0}^{\infty} \sum_{n=0, m\neq n}^{\infty} \frac{a_m \bar{b}_n}{m-n} \right|^2 < \pi^2 \|a\|^2 \|b\|^2 . \quad (3.11)$$

The inequalities (3.10) and (3.11) are refinements of the Hilbert-Ingham inequality

$$\left|\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m\bar{b}_n}{m+n+\frac{1}{2}}\right|\leq\pi\left\|a\right\|\left\|b\right\|.$$

We have now a new inequality according to Theorem 3.1 (ii).

Theorem 3.2. *With the assumptions as Theorem 3.1, if $\lambda=\frac{1}{4}$, then*

$$\left|\frac{1}{2\pi}U_2\left(a,b\right)\right|^2+\left|V\left(a,b\right)\right|^2\leq\pi^2\left\|a\right\|^2\left\|b\right\|^2-\frac{1}{\pi^2}r\left(a\right)r\left(b\right). \tag{3.12}$$

4. Applications

In this section we assume that $u(n)=Z_n+\frac{\lambda}{2}$, and λ is an integer. We shall give some applications on Corollary 3.1.

Let $f\left(z\right)=\sum_{n=0}^{\infty}a_nz^n\in H_p$ with $p>0$ and $f\left(z\right)$ be analytic in the unit disc $|z|<1$. Then

$$\int_{-1}^{1}\left|f\left(t\right)\right|^p dt\leq\frac{1}{2}\int_{-\pi}^{\pi}\left|f\left(e^{i\theta}\right)\right|^p d\theta, \tag{4.1}$$

where the coefficient $\frac{1}{2}$ is the best possible. This is the famous Fejer-Riesz inequality in H_p function (see [4]).

Notice that $\int_0^1|f(t)|^p dt\leq\int_{-1}^1|f(t)|^p dt$ and $\int_{-\pi}^{\pi}\left|f\left(e^{i\theta}\right)\right|^p d\theta=\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^p d\theta$, hereby we have the following Fejer-Riesz type inequality:

$$\int_{0}^{1}\left|f\left(t\right)\right|^p dt\leq\frac{1}{2}\int_{0}^{2\pi}\left|f\left(e^{i\theta}\right)\right|^p d\theta. \tag{4.2}$$

We will give a refinement of (4.2) in what follows.

Theorem 4.1. Let $f\left(z\right)=\sum_{n=0}^{\infty}a_nz^{u(n)-\frac{1}{2}}\in H_p$ with $p>0$ and $f\left(z\right)$ be analytic in the unit disc $|z|<1$. Then

$$\left|\int_{0}^{1}\left|f\left(t\right)\right|^p dt\right|^2+\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}t\left|f\left(-e^{i\theta}\right)\right|^p d\theta\right|^2\leq\left(\frac{1}{2}\int_{0}^{\pi}\left|f\left(e^{i\theta}\right)\right|^p d\theta\right)^2. \tag{4.3}$$

Proof. At first, we prove the theorem for case $p=2$. Since $f\left(z\right)=$

$\sum\limits_{m=0}^{\infty} a_m z^{u(m)-\frac{1}{2}}$, we have

$$\int\limits_0^1 |f(t)|^2\,dt = \sum_{m=0}^{\infty}\ \sum_{n=0,\ u(m)+u(n)\neq 0}^{\infty} \frac{a_m\overline{a_n}}{u(m)+u(n)} = U(a),$$

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \left|f\left(e^{it}\right)\right|^2 dt = \|a\|^2 .$$

By Lemma 2.3, we have

$$\left|\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} t\left|f\left(-e^{it}\right)\right|^2 dt\right| = \left|\sum\nolimits_{m=0}^{\infty}\ \sum_{n=0,\ u(m)\neq u(n)}^{\infty} \frac{a_m\overline{a_n}}{u(m)-u(n)}\right| = |V(a)| .$$

According (3.8), we have

$$\left|\int\limits_0^1 |f(t)|^2\,dt\right|^2 + \left|\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} t\left|f\left(-e^{it}\right)\right|^2 dt\right|^2 \le \left(\frac{1}{2}\int\limits_0^{2\pi} \left|f\left(e^{it}\right)\right|^2 dt\right)^2 .$$

Hence the inequality (4.3) is valid when $p=2$.

If $p \neq 2$, by the Blaschke decomposition theorem we have $f(z) = B(z)\,G(z)$, where $B(z)$ is the Blaschke function and $G(z) \neq 0$. $|B(z)| \le 1$ in $|z| < 1$ and $\left|B\left(e^{it}\right)\right| = 1$. Let $F(z) = (G(z))^{p/2} \in H_2$. According to the above result for $p=2$, we have

$$\left|\int\limits_0^1 |f(t)|^p\,dt\right|^2 + \left|\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} t\left|f\left(-e^{it}\right)\right|^p dt\right|^2 \left|\int\limits_0^1 |F(t)|^2\,dt\right|^2 + \left|\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} t\left|F\left(-e^{it}\right)\right|^2 dt\right|^2$$

$$\le \left\{\frac{1}{2}\int\limits_0^{2\pi} \left|F\left(e^{it}\right)\right|^2 dt\right\}^2 = \left\{\frac{1}{2}\int\limits_0^{2\pi} \left|G\left(e^{it}\right)\right|^p dt\right\}^2 = \left\{\frac{1}{2}\int\limits_0^{2\pi} \left|f\left(e^{it}\right)\right|^p dt\right\}^2 .$$

The proof of the theorem is finished. □

Let $f(z) = \sum_{m=0}^{\infty} c_m z^m \in H_1$. Then

$$\sum_{n=0}^{\infty} \frac{|c_n|}{n+1} \le \frac{1}{2}\int_0^{2\pi} \left|f\left(e^{it}\right)\right| dt. \tag{4.4}$$

This is the Hardy type inequality in H_p function (see [4]). We will give a refinement of (4.4) as follows.

Theorem 4.2. *Let $f(z) = \sum_{m=0}^{\infty} c_m z^{u(m)}$ be analytic in the unit disc $|z| < 1$ and $f \in H_1$. Then*

$$\left(\sum_{n=0}^{\infty} \frac{|c_n|}{u(n)}\right)^2 + \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} t\left|f\left(-e^{it}\right)\right|^2 dt\right|^2 \le \left(\frac{1}{2}\int_0^{2\pi} \left|f\left(e^{it}\right)\right| dt\right)^2 \tag{4.5}$$

Proof. By Blaschke Decomposition Theorem we have

$$f(z) = B(z)\, G(z) = B(z)\, G^{\frac{1}{2}}(z)\, G^{\frac{1}{2}}(z) = f_1(z)\, f_2(z),$$

where $B(z)$ is Blaschke function and $f_1, f_2 \in H_2$. $f_1(z) = B(z)\, G^{\frac{1}{2}}(z) = \sum_{m=0}^{\infty} a_m z^{u(m)}$, $f_2(z) = G^{\frac{1}{2}}(z) = \sum_{n=0}^{\infty} b_n z^{u(n)}$. It is easy deduce that

$$\begin{aligned}\|a\|^2 = \|b\|^2 = \sum_{m=0}^{\infty} |a_m|^2 = \sum_{n=0}^{\infty} |b_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G^{\frac{1}{2}}\left(e^{it}\right)\right|^2 dt \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|B\left(e^{it}\right) G^{\frac{1}{2}}\left(e^{it}\right)\right|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(e^{it}\right)\right| dt.\end{aligned} \tag{4.6}$$

It is easy to deduce that

$$\sum_{n=0}^{\infty} \frac{|c_n|}{u(n)} \le \sum_{n=0}^{\infty} \sum_{r+s=n} \frac{|a_r|\,|a_s|}{u(r)+u(s)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{|a_m|\,|a_n|}{u(m)+u(n)} = U(a). \tag{4.7}$$

By Lemma 2.3, we find that

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} t\left|f\left(-e^{it}\right)\right| dt\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} t\left|f_1\left(-e^{it}\right)\right|^2 dt\right| \\ = \left|\sum_{m=0}^{\infty} \sum_{n=0,\ u(m)\neq u(n)}^{\infty} \frac{|a_m|\,|a_n|}{u(m)-u(n)}\right| = V(a).\end{aligned} \tag{4.8}$$

It follows from (4.6), (4.7), (4.8) and Corollary 3.1 that the inequality (4.5) is valid. □

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