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EXISTENCE AND UNIQUENESS OF THE OPTIMAL CONTROL FOR SOME SYSTEMS WITH SMALL PARAMETER AND TIME LAG

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Abstract: The present article considers the problem of transfer for a dynamic controlled system with small parameter and time lag from a given initial position to a given final one at quadratical criteria. A method for the solution was proposed, based on the perturbation theory analysis. The uniqueness of the solution for the maximum principle was proved.

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1. Introduction and Problem Statement

Time delays are frequently encountered in the behavior of many physical processes and very often are the main cause for poor performance and instability of control systems. The significant growth of interest in such systems is due to their various applications in control theory and automatic regulation [4], biology and ecology [2] and [3], biomedicine [1] and others. In view of this, time delay systems is a topic of great practical importance which attracted a great deal of interest for several decades; see [8]. The investigation of systems with time lag is associated with a number of specific singularities. One of them is the absence of simple expressions for the translation operator along the system trajectories. In view of this, it is difficult to extend some results of the control theory of ordinary equations to systems with time lag.

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The purpose of this paper is to present an approximate method for the optimal control synthesis for systems of the form:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \epsilon f(t,\, x(t-h)),$$
$$0 \le t \le T, \quad x(0) = a_0, \quad x(\theta) = \phi(\theta), \quad \theta \in [-h,\, 0), \tag{1}$$

where x is a coordinate vector from the Euclidean space $\mathbf{R}^n$, u is a control vector from $\mathbf{R}^m$, and matrices A, B have continuous and bounded elements. The small parameter $\epsilon \ge 0$, constant time lag $h > 0$, vector a_0 and final time T are given. The function f is continuous with respect to all its variables, continuously differentiable with respect to its second argument, and for some constants C_1, $C_2 \ge 0$, satisfies the condition

$$|f(t,\, x(t-h))| \le C_1 + C_2|x(t-h)|^2, \tag{2}$$

where $|\cdot|$ is the Euclidean norm. The continuous function ϕ is known.

Occasionally we shall designate the solution of (1) by $x(t,\, u)$ to emphasize its dependence on the control u. We now state our problem.

Problem 1. Find the control vector u which translates the initial system from the initial state a_0 to the final state a_1 such that the criterion

$$J(u) = \int_0^T |u(s)|^2\, ds \tag{3}$$

is minimized.

This means that $x(0,\, u) = a_0$ and $x(T,\, u) = a_1$. We introduce the notation

$$z(t,\, s) = \exp \int_s^t A(\tau)\, d\tau, \quad G_1(t) = \int_0^t z(t,\, s)B(s)B'(s)z'(t,\, s)\, ds\, G^{-1} \tag{4}$$

and

$$G = \int_0^T z(T,\, s)B(s)B'(s)z'(T,\, s)\, ds\,, \tag{5}$$

where the prime denotes transpose.

If G is a nonsingular matrix, then with $\epsilon = 0$, Problem 1 has an explicit analytical solution [6]. We assume throughout this paper that G is nonsingular. With $\epsilon \ne 0$ and $h = 0$ Problem 1 was also investigated in [6] under the following conditions: f is an analytic function of x, zero-approximation system is completely controllable, and the minimal function satisfies certain condition.

In the present work we present a new approach to Problem 1 with $\epsilon \ne 0$ and $h \ne 0$ under the above assumptions concerning the parameters of the system (1) to construct sequences of controls and trajectories in such a way as to approximate the optimal ones, and to obtain an error estimate and an upper bound of the parameter ϵ for which the proposed method is correct.

2. Existence

Theorem 1. *Under assumptions on the coefficients of the system (1) and (15) the existence of the optimal control in Problem 1 is guaranteed.*

Proof. We start by proving the existence of an *admissible control*, i.e. a control $u(t)$ steering the system (1) from a_0 to a_1 at time T, so that $J(u) < +\infty$. Here we use some results of [9] concerning to perturbation theory.

Let $v_0(t)$ and $x_0(t)$ be the solution of Problem 1 for the system (1) with $\epsilon = 0$. Then

$$\begin{aligned} v_0(t) &= B'(t)z'(T,\, t)G^{-1}(a_1 - z(T,\, 0)a_0), \\ x_0(t) &= z(t,\, 0)a_0 + G_1(t)(a_1 - z(T,\, 0)a_0). \end{aligned} \tag{6}$$

Consider a sequence of controls v_k as a sequence solving Problem 1 for the system

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)v(t) + \epsilon f(t,\, x_{k-1}(t-h)), \quad k = 1, 2, \dots, \\ &x(0) = a_0, \quad 0 \le t \le T, \quad x_k(\theta) = \phi(\theta), \quad \theta \in [-h,\, 0). \end{aligned} \tag{7}$$

Since G is a nonsingular matrix, the sequence v_k exists. We denote by x_k a trajectory of the system (7) corresponding to the control v_k. Using Lagrange's multipliers method [7, §1.1], we obtain that for $k \ge 1$

$$\begin{aligned} &v_k(t) = B'(t)z'(T,\, t)G^{-1}\lambda_k, \\ &x_k(t) = z(t,\, 0)a_0 + G_1(t)\lambda_k + \epsilon \int_0^t z(t,\, s)f(s,\, x_{k-1}(s-h))\, ds, \\ &\lambda_k = a_1 - z(T,\, 0)a_0 - \epsilon \int_0^T z(T,\, s)f(s,\, x_{k-1}(s-h))\, ds. \end{aligned} \tag{8}$$

When $k = 0$, the quantities v_0, x_0, λ_0 are also given by (8) with $\epsilon = 0$. Let m_0 be

$$m_0 = \max_{\substack{0 \le t \le T, \\ -h \le \theta < 0}} \Big(|z(t,\, 0)a_0 + \lambda_0 G_1(t)|,\, |\phi(\theta)| \Big). \tag{9}$$

Hence and from (6) it follows that

$$|x_0(t)| \le m_0, \;\; -h \le t \le T. \tag{10}$$

Now we rewrite the formula for $x_k(t)$ in the form

$$x_k(t) = x_0(t) + \epsilon \int_0^t z(t,\, s)f(s,\, x_{k-1}(s-h))\, ds$$

$$-\epsilon\int_0^T z(T,\,s)f(s,\,x_{k-1}(s-h))\,ds\cdot G_1(t). \quad (11)$$

Let m_1 be given by

$$m_1=\max_{0\le t\le T}\left\{\int_0^t ||z(t,\,s)||ds+\int_0^T ||z(T,\,s)||ds\cdot||G_1(t)||\right\}, \quad (12)$$

where $||\cdot||$ denotes the Euclidean matrix norm. Fix some number β such that

$$\beta>(C_1+C_2m_0^2)m_1. \quad (13)$$

From (11), (13) and (2) we find that

$$|x_k(t)|\le m_0+\epsilon\beta,\ \ -h\le t\le T, \quad (14)$$

where (14) is valid for all values of ϵ satisfying the inequality

$$(C_1+C_2(m_0+\epsilon\beta)^2)m_1\le\beta. \quad (15)$$

Hence, if (15) holds, the sequence $x_k(t)$ is uniformly bounded by virtue of (14). But because of (8), the sequence $v_k(t)$ is also uniformly bounded. Hence, in the light of (1) and (2), the sequence $x_k(t)$ is equicontinuous on the segment $[-h,\,T]$. It follows from Arzela's Theorem [5, p. 110] that the sequence $x_k(t)$ is compact in the space of continuous function on the segment $[-h,\,T]$. Without loss of generality we consider the compact sequences as convergent in the appropriate metric.

Denote by x and v the uniform limits of the sequences $x_k(t)$ and $v_k(t)$, respectively. According to the construction, we have $x(0)=a_0$, $x(T)=a_1$ and because of (7), (8) a pair $x(t)$, $v(t)$ satisfies the equation (1). Finally, note that the boundedness of $v(t)$ follows from the uniform boundedness of v_k, and so $J(v)\le m_2$, where

$$m_2=\sup_{0\le t\le T}\Bigg(||B'(t)z'(T,\,t)||\Bigg(|a_1-z(T,\,0)a_0| + \epsilon(C_1+C_2(m_0+\epsilon\beta)^2)\int_0^T ||z(T,\,s)||\,ds\Bigg)\Bigg). \quad (16)$$

Thus, the existence of an admissible control is established. □

Theorem 2. *The existence of an admissible control implies the existence of an optimal one.*

Proof. We introduce, as in [10], a minimizing sequence of controls $u_i(t)$ as follows:

$$J_0=\inf J(u)=\lim_{i\to\infty}J(u_i), \quad (17)$$

where the infimum is taken over the set of admissible controls. Without loss of generality we assume that

$$J(u_i) \leq m_2 \tag{18}$$

is satisfied for all i. Note that $x(0, u_i) = a_0$ and $x(T, u_i) = a_1$, since all u_i are admissible controls. From this and [7, §4.1] the uniform boundedness of the sequence $x(t, u_i)$ on the segment $[-h, T]$ follows.

From the uniform boundedness of the sequence $x(t, u_i)$ and (18), we obtain that $x(t, u_i)$ uniformly converges to $x(t)$ as $t \to \infty$. By virtue of (18) the sequence $u_i(t)$ is weakly compact in the space of square integrable functions on the segment $[0, T]$. Hence, we assume that $u_i(t)$ weakly converges to $u(t)$ as $i \to \infty$. From the uniform convergence of $x(t, u_i)$ to $x(t)$ and from the initial equation (1) we conclude that $x(t) = x(t, u)$. Hence, the pair $x(t)$, $u(t)$ satisfies the system (1), and also, by the construction, $x(0) = a_0$ and $x(T) = a_1$. It only remains to show that $J(u) = J_0$. The weak convergence of $u_i(t)$ to $u(t)$ implies [5, p. 194] that

$$J(u) \leq \lim_{i\to\infty} J(u_i). \tag{19}$$

Moreover, as noted above, the control $u(t)$ is the admissible control. Hence, from the definition of J_0 and (19) it follows that $J_0 = J(u)$. Therefore, the existence of the optimal control for all values of ϵ satisfying (15), is proved. □

It now follows from Pontryagin's maximum principle [10, p. 233] that the optimal pair $x(t)$, $u(t)$ is a solution of the boundary value problem consisting of the system (1) and the relations

$$\begin{aligned} &x(0) = a_0, \quad x(T) = a_1, \quad u(t) = \frac{1}{2}B'(t)\psi(t), \\ &\dot{\psi}(t) = -A'(t)\psi(t) - \epsilon f_1(t+h, x(t))\psi(t+h), \\ &\psi(s) \equiv 0, \quad s > T. \end{aligned} \tag{20}$$

Here $\psi(t)$ is a vector of the conjugate variables of the maximum principle, and a square matrix f_1 as i-tuple has partial derivatives of $f(t, x(t-h))$ with respect to $x_i(t-h)$.

We assume that $f_1(t+h, x(t))$ satisfies a local Lipschitz condition of a type (2). More precisely, we assume that for $x(t) \in \mathbf{R}^n$ and some constants C_3, C_4, $C_5 \geq 0$

$$|f_1(t+h, x(t))| \leq C_3 + C_4|x(t)|^2, \tag{21}$$

and for all $x_1(t)$, $x_2(t)$ such that $|x_1(t)| \leq N$, $|x_2(t)| \leq N$,

$$|f_1(t+h, x_1(t)) - f_1(t+h, x_2(t))| \leq C_5(N)|x_1(t) - x_2(t)|. \tag{22}$$

Later, constants such as the constants m_0, m_1, m_2 will occur frequently depending on the coefficients of the system (1). For purposes of brevity we reduce the explicit form of these constants, restricting ourselves to the method in which they are obtained.

The construction of the optimal trajectory described above implies the existence of a constant m_3, depending only on the coefficients of (1), such that

$$|x(t)| \leq m_3, \;\; -h \leq t \leq T. \tag{23}$$

We now show that the conjugate variable $\psi(t)$ is also bounded. Relations (1) and (20) imply that

$$\begin{aligned}\psi(t) = -\epsilon \int_0^t z'(t,\, s)^{-1} f_1(s+h,\, x(s))\psi(s+h)\, ds \\ + 2z'(T,\, t)G^{-1}\biggl[a_1 - z(T,\, 0)a_0 - \epsilon \int_0^T z(T,\, s) f(s,\, x(s-h))\, ds \\ + \frac{\epsilon}{2}\int_0^T z(T,\, s)B_1(s)\int_0^s z'(s,\, r)^{-1} f_1(r+h,\, x(r))\psi(r+h)\, dr\, ds\biggr],\end{aligned} \tag{24}$$

where $B_1(t) = B(t)B'(t)$. The relation (24) is the Fredholm integral equation of the second kind, containing the small parameter ϵ. It is known that such an equation has one and only one solution for all ϵ [5, p. 467], and that this solution may be represented as a series of the powers of ϵ. Moreover, convergence of that series is provided by the choice of ϵ. Therefore, there exists a constant m_4 such that

$$|\psi(t)| \leq m_4, \;\; 0 \leq t \leq T. \qquad \square \tag{25}$$

3. Convergence and Uniqueness of the Solution

In this section we prove the convergence of the consistent approximations of the perturbation method of solving (1), (20). Our approximations $x_k(t)$ and $\psi_k(t)$ are defined to be the solution of the following boundary-value problem

$$\begin{aligned}&\dot{x}_k(t) = A(t)x_k(t) + B(t)u_k(t) + \epsilon f(t,\, x_{k-1}(t-h)),\\ &x_k(0) = a_0, \;\; x_k(T) = a_1, \;\; u_k(t) = \frac{1}{2}B'(t)\psi_k(t),\\ &0 \leq t \leq T, \;\; x_k(\theta) = \phi(\theta), \;\; \theta \in [-h,\, 0)\end{aligned} \tag{26}$$

and

$$\begin{aligned}&\dot{\psi}_k(t) = -A'(t)\psi_k(t) - \epsilon f_1(t+h,\, x_{k-1}(t))\psi_{k-1}(t+h),\\ &\psi_k(s) \equiv 0, \;\; s > T.\end{aligned} \tag{27}$$

Null approximations $x_0(t)$ and $\psi_0(t)$ are also determined by (26) and (27) with $\epsilon = 0$.

Theorem 3. *Successive approximations determined in (26), (27) uniformly converge to the solution of the boundary-value problem (1), (20).*

Proof. The proof of this theorem is similar to the proof of the existence of the admissible control in Problem 1. First, we establish the uniform boundedness of sequences x_k and ψ_k. From (26), (27) and the non-singularity of G we obtain that the initial condition for (27) is

$$\psi_k(0) = 2z'(T,\, 0)G^{-1}\left[a_1 - z(T,\, 0)a_0 - \epsilon\int_0^T z(T,\, s)f(s,\, x_{k-1}(s-h))\,ds\right.$$
$$\left. + \frac{\epsilon}{2}\int_0^T z(T,\, s)B_1(s)\int_0^s z'(s,\, r)^{-1}f_1(r+h,\, x_{k-1}(r))\psi_{k-1}(r+h)\,dr\,ds\right]. \quad (28)$$

We now use (26)-(28) to derive further explicit representations of the consistent approximations x_k and ψ_k

$$x_k(t) = z(t,\, 0)a_0 + \int_0^t z(t,\, s)\left[\frac{1}{2}B_1(s)\psi_k(s) + \epsilon f(s,\, x_{k-1}(s-h))\right]ds,$$
$$\psi_k(t) = z'(t,\, 0)^{-1}\psi_k(0) \quad (29)$$
$$- \epsilon\int_0^t z'(t,\, s)^{-1}f_1(s+h,\, x_{k-1}(s))\psi_{k-1}(s+h)\,ds.$$

Note that $x_0(t)$ and $\psi_0(t)$ are also given by (29) with $\epsilon = 0$. Further arguments necessary for the estimate justification such as (14) and (25), are similar to that used in Section 2; the only change is that instead of (8), we employ (29). Therefore, we do not write an explicit form of constants n_0, n_1 having the same sense as m_0, m_1 from Section 2, but only indicate the method for obtaining them from (29). First, using the null approximation $x_0(t)$, $\psi_0(t)$, we choose n_0 so that

$$n_0 \geq \max_{\substack{0\leq t\leq T,\\ -h\leq r<T}}\Big(|x_0(r)|,\, |\psi_0(t)|\Big). \quad (30)$$

Next, fix some positive number σ such that

$$\sigma > (C_1 + C_2 n_0^2)n_1 + (C_3 + C_4 n_0^2)n_1 n_0, \quad (31)$$

where the constant n_1 is determined from (29) in the same way as m_1 was determined from (8). As in our derivation of (14), we conclude that

$$\max_{\substack{0\leq t\leq T,\\ -h\leq r<T}}\Big(|x_k(r)|,\, |\psi_k(t)|\Big) \leq n_0 + \epsilon\sigma \quad (32)$$

for all values of ϵ, satisfying

$$(C_1 + C_2(n_0 + \epsilon\sigma)^2)n_1 + (C_3 + C_4(n_0 + \epsilon\sigma)^2)(n_0 + \epsilon\sigma)n_1 \leq \sigma. \tag{33}$$

From the uniform boundedness of $x_k(t)$ and $\psi_k(t)$, and by repeating the corresponding arguments of Section 2, we obtain that the sequences x_k and ψ_k uniformly converge to x and ψ, respectively, as $k \to \infty$, which satisfies the boundary value problem (1), (20). □

We have constructed two solutions of the boundary value problem (1), (20): first in Section 2, and second, using the perturbation method, in this section. Now define a constant m_5

$$m_5 = \max\{m_3,\, m_4,\, n_0 + \epsilon\sigma\} \tag{34}$$

and prove

Theorem 4. *If ϵ is sufficiently small and $|x| \leq m_5$, then (1), (20) has only a single solution.*

Proof. Let us assume the contrary. Let x_0, ψ_0 and x_1, ψ_1 be two solutions to the problem (1), (20). Note that in the domain $|x| \leq m_5$ the function $f(t,\, x(t-h))$ clearly satisfies the Lipschitz condition with a constant L determined by the upper bound of the function $f_1(t+h,\, x(t))$ in the indicated domain. Using this remark, we investigate the differences $x_1(t) - x_0(t)$ and $\psi_1(t) - \psi_0(t)$ obtained from (29) with $k = 0$, 1 and (24). It is easily seen that the terms not containing ϵ cancel and after some simplifications we obtain

$$|x_1(t) - x_0(t)| + |\psi_1(t) - \psi_0(t)| \leq \epsilon m_6(|x_1(t) - x_0(t)| + |\psi_1(t) - \psi_0(t)|). \tag{35}$$

Here the constant m_6 can be easily estimated in terms of the parameters of (1). To ensure the desired uniqueness of a solution of (1), (20), it suffices to require that

$$\epsilon m_6 < 1. \tag{36}$$

With the help of (28) and (29) analogously to (35) we obtain

$$|x_{k+1}(t) - x_k(t)| + |\psi_{k+1}(t) - \psi_k(t)| \leq \epsilon m_6(|x_k(t) - x_{k-1}(t)| + |\psi_k(t) - \psi_{k-1}(t)|), \tag{37}$$

where functions $x_k(t)$ and $\psi_k(t)$ are given by (28) and (29). From this it follows that if $x(t)$, $\psi(t)$ is the solution to the boundary value problem (1), (20), then

$$|x(s) - x_k(s)| + |\psi(t) - \psi_k(t)| \frac{1}{1-\epsilon m_6}(\epsilon m_6)^k \big[|x_1(s) - x_0(s)| + |\psi_1(t) - \psi_0(t)|\big] \leq \frac{1}{1-\epsilon m_6}(\epsilon m_6)^k \epsilon m_7, \quad -h \leq s \leq T,\ 0 \leq t \leq T, \tag{38}$$

where the last passage in (38) was realized with the help of (29) when $k = 0, 1$. $\square$

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